

PROPERTIES OF SOME INTEGRAL TRANSFORMS ARISING IN
TOMOGRAPHY

A Dissertation

by

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ABSTRACT

This dissertation deals with several types of imaging: radio tomography, single scattering optical tomography, photoacoustic tomography, and Compton camera imaging. Each of these tomographic techniques leads to a Radon-type transform: radio tomography brings about an elliptical Radon transform, single scattering optical tomography reduces to the V-line Radon transform, and photoacoustic tomography with line detectors boils down to a cylindrical Radon transform. We also introduce a different Radon-type transform arising in photoacoustic tomography with circular detectors, and study mathematically similar object, a toroidal Radon transform. We also consider the cone transform arising in Compton camera imaging as well as the windowed ray transform.

We provide inversion formulas for all these transforms. When given some Radon-type transform, we are interested not only in inversion formulas, but also in range conditions, and stability. We thus address range conditions, a stability estimate for some of the Radon-type transforms above.

DEDICATION

To my parents, Tae Yul Mun and Kyung Ae Park

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1. INTRODUCTION

Tomography aims to find the internal information of a non-transparent object by sending some signals through it. Electromagnetic waves of various frequencies such as radio, microwaves, visual light, X-rays, γ -rays, as well as acoustic waves are common. Computed tomography obtains an image by mathematical processing of measured data. In the simplest cases, this amounts to reconstructing a function from its line or plane integrals. Radon transform, an integral transform that maps a given function into its integrals over hyperplanes, was introduced in 1917 by J. Radon. In 1938, the X-ray transform, defined by integrating over lines rather than hyperplanes, was introduced by F. John [39]. This transform is very closely related to the Radon transform.

Other types of tomography have been introduced that also lead to Radon-type transforms. For example, thermoacoustic tomography or radar and sonar imaging leads to a spherical Radon transform which maps a given function onto its integrals over spheres.

In this dissertation, we consider several of these types of tomography and study Radon-type transforms modeling them.

This dissertation is organized as follows. Before studying Radon-type transform, we introduce some basic properties of the Radon transform in subsection 1.1, as we will need them later. Section 2 is devoted to Radio tomography and an elliptical Radon transform. We provide not only inversion formulas, but also local uniqueness and a stability estimate.

In section 3, we study single scattering optical tomography and the V-line Radon transform on a disk. An inversion formula is derived.

Photoacoustic Tomography (PAT) with line and circular detectors is considered in section 4. We obtain inversion formulas, support theorem, a stability estimate, and range conditions for cylindrical Radon transforms arising in PAT with line detectors. We suggest a Radon-type transform arising in PAT with circular detectors and study it.

A toroidal Radon transform is studied in section 5.

Section 6 is devoted to Compton camera imaging and the cone transform arising in it. Inversion formulas using full data for 2 and 3 dimensional cases are obtained. We also study some properties of the cone transform with a fixed central axis.

Lastly, the windowed ray transform is studied in section 7.

1.1 Definitions and properties of some integral transforms

In this subsection, we introduce some transforms and study their basic properties that will be needed later.

Let $f(x)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, the function space of functions all of whose derivatives are rapidly decreasing. The **Fourier transform** of f is defined as

$$\hat{f}(\xi) := \mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^n,$$

the **inverse Fourier transform** is

$$f(x) = \mathcal{F}^{-1}\hat{f}(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} d\xi.$$

The Plancherel theorem claims that for $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, one has

$$\int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\xi) d\xi = 2\pi \int_{\mathbb{R}^n} f(x) g(x) dx. \quad (1.1)$$

The Fourier transform extends to a bijection from $L^2(\mathbb{R}^n)$ to itself.

The **Hankel transform** of order k of a function $f \in \mathcal{S}(\mathbb{R})$ is given by

$$H_k f(\eta) = \int_0^\infty f(r) J_k(\eta r) r dr,$$

where J_k is the Bessel function of the first kind of order k with $k \leq -1/2$. The inverse Hankel transform of $H_k f(\eta)$ is defined as

$$f(r) = \int_0^\infty H_k f(\eta) J_k(\eta r) \eta d\eta.$$

The version of the Plancherel theorem for the Hankel transform states that

$$\int_0^\infty f(r) g(r) r dr = \int_0^\infty H_k f(\eta) H_k g(\eta) \eta d\eta. \quad (1.2)$$

The **Mellin transform** \mathcal{M} is an integral transform on $(0, \infty)$ which is defined by

$$\mathcal{M}f(s) := \int_0^\infty f(x) x^{s-1} dx,$$

and the inverse Mellin transform \mathcal{M}^{-1} is

$$\mathcal{M}^{-1}f(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} f(s) ds.$$

It is known [25, 70] that

$$\mathcal{M}(xf)(s) = \mathcal{M}f(s+1) \text{ and } \mathcal{M}\left[\int_x^\infty f(t) dt\right](s) = \mathcal{M}f(s+1)/2. \quad (1.3)$$

Also, we have $\mathcal{M}(f \times g)(s) = \mathcal{M}f(s)\mathcal{M}g(s)$ where

$$f \times g(s) := \int_0^\infty f(r)g\left(\frac{s}{r}\right) \frac{dr}{r}.$$

One of the most important for computed tomography integral transforms is the **Radon transform**, which we denote by \mathcal{R} . The Radon transform assigns to a given function $f \in \mathcal{S}(\mathbb{R}^n)$ its integrals over the hyperplanes, i.e.

$$\mathcal{R}f(\theta, s) = \int_{x \cdot \theta = s} f(x) dx \quad \text{for } (\theta, s) \in S^{n-1} \times \mathbb{R}.$$

The Radon transform has the following properties:

1. The function $\mathcal{R}f(\theta, s)$ is even, that is, $\mathcal{R}f(\theta, s) = \mathcal{R}f(-\theta, -s)$.
2. The **Fourier slice theorem** states that $\widehat{\mathcal{R}f}(\theta, \sigma) = \hat{f}(\sigma\theta)$ for $f \in \mathcal{S}(\mathbb{R}^n)$.

Here

$$\widehat{\mathcal{R}f}(\theta, \sigma) = \int_{\mathbb{R}} \mathcal{R}f(\theta, s) e^{-is\sigma} ds.$$

3. The following statement is called the **support theorem**: Let $f \in \mathcal{S}(\mathbb{R}^n)$ and K be a convex compact set in \mathbb{R}^n . If $\mathcal{R}f(\theta, s) = 0$ for every plane $x \cdot \theta = s$ not meeting K , then $f = 0$ outside K .
4. Let B^n be the unit ball in \mathbb{R}^n . For each α , there exist positive constants $c(\alpha, n)$ and $C(\alpha, n)$ such that for any smooth function f with compact support in B^n ,

$$c(\alpha, n) \|f\|_\alpha \leq \|\mathcal{R}f\|_{\alpha+(n-1)/2} \leq C(\alpha, n) \|f\|_\alpha, \quad (1.4)$$

where

$$||f||_\alpha^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^\alpha |\hat{f}(\xi)|^2 d\xi$$

and

$$||g||_\alpha^2 = \int_{S^{n-1}} \int_{\mathbb{R}} (1 + \sigma^2)^\alpha |\hat{g}(\theta, \sigma)|^2 d\sigma d\theta,$$

where \hat{g} is the Fourier transform of a function g on $S^{n-1} \times \mathbb{R}$.

5. We have the following inversion formulas: For $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha < n$,

$$f = 2^{-1} (2\pi)^{1-n} I^{-\alpha} \mathcal{R}^\# I^{\alpha-n+1} \mathcal{R} f,$$

where the Riesz potential I^α is defined by

$$\widehat{I^\alpha f}(\xi) = |\xi|^{-\alpha} \hat{f}(\xi)$$

and the backprojection, the dual operator to \mathcal{R} , is given by

$$\mathcal{R}^\# g(x) := \int_{S^{n-1}} g(\theta, x \cdot \theta) d\theta.$$

Here g is a function on $S^{n-1} \times \mathbb{R}$.

6. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We expanded f and $g = \mathcal{R}f$ in spherical harmonics Y_{lk} , i.e.,

$$f(x) = \sum_{l=0}^{\infty} \sum_{k=0}^{N(n,l)} f_{lk}(|x|) Y_{lk}(x/|x|), \quad g(\theta, s) = \sum_{l=0}^{\infty} \sum_{k=0}^{N(n,l)} g_{lk}(s) Y_{lk}(\theta),$$

$$N(n, l) = \frac{(2l+n-2)(n+l-3)!}{l!(n-2)!}, \quad N(n, 0) = 1.$$

Then we have for $s > 0$

$$g_{lk}(s) = |S^{n-2}| \int_s^\infty C_l^{(n-2)/2} \left(\frac{s}{r} \right) \left(1 - \frac{s^2}{r^2} \right)^{(n-3)/2} f_{lk}(r) r^{n-2} dr,$$

where $C_l^{(n-2)/2}$ is the (normalized) Gegenbauer polynomial of degree l and $|S^n|$ is an surface area of a unit sphere S^n .

From the above relation, one obtains the following inversion formula:

$$\begin{aligned} f_{lk}(r) &= c(n) \int_r^\infty (s^2 - r^2)^{(n-3)/2} C_l^{(n-2)/2} \left(\frac{s}{r} \right) g_{lk}^{(n-1)}(s) ds, \\ c(n) &= \frac{(-1)^{n-1}}{2\pi^{n/2}} \frac{\Gamma((n-2)/2)}{\Gamma(n-2)} \end{aligned} \quad (1.5)$$

(for $n = 2$ one has to take the limit $n \rightarrow 2$, i.e. $c(2) = -1/\pi$).

Lastly, the X -ray and divergent beam transforms are defined by

$$\begin{aligned} Pf(u, \theta) &= \int_{\mathbb{R}} f(u + t\theta) dt, (u, \theta) \in \mathbb{R}^n \times S^{n-1}, \\ Df(u, \theta) &= \int_0^\infty f(u + t\theta) dt, (u, \theta) \in \mathbb{R}^n \times S^{n-1}. \end{aligned}$$

That is, $Pf(u, \theta)$ is the integral of f along the line through $u \in \mathbb{R}^n$ in the direction of $\theta \in S^{n-1}$ and $Df(u, \theta)$ is the integral of f along the half line starting at the point $u \in \mathbb{R}^n$ in the direction of $\theta \in S^{n-1}$.

2. RADIO TOMOGRAPHY AND AN ELLIPTICAL RADON TRANSFORM*

Radon-type transforms that integrate functions over various sets of ellipses/ellipsoids have been arising in recent decade, due to studies in synthetic aperture radar (SAR) [4, 14, 44, 45], ultrasound reflection tomography [3, 30], and radio tomography [75, 76, 77]. In particular, radio tomography is a new imaging method, which uses a wireless network of radio transmitters and receivers to image the distribution of attenuation within the network. The usage of radio frequencies brings in significant non-line-of-sight propagation, since waves propagate along many paths from a transmitter to a receiver. Given a transmitter and a receiver, wave paths observed for a given duration are all contained in an ellipsoid with foci at these two devices. It was thus suggested in [75, 76, 77] to approximate the obtained signal by the volume integral of the attenuation over this ellipsoid, which is the model we study in this section.

Due to these applications, there have been several papers devoted to such “elliptical Radon transforms.” The family of ellipses with one focus fixed at the origin and the other one moving along a given line was considered in [45]. In the same paper, the family of ellipses with a fixed focal distance was also studied. The authors of [3, 30] dealt with the case of circular acquisition, when the two foci of ellipses with a given focal distance are located on a given circle. A family of ellipses with two moving foci was also handled in [14].

In all these works, however, the ellipses have varying eccentricity. Also, their data were the line integrals of the function over ellipses rather than area integrals. The radio tomography application makes it interesting to consider integrals over solid

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ellipsoids. In this article, we consider the volume integrals of an unknown attenuation function over the family of ellipsoids of rotation in \mathbb{R}^n with a fixed eccentricity and two foci located in a given hyperplane. We thus reserve the name **elliptical Radon transform** $R_E f$ for the volume integral of a function f over this family of ellipsoids.

The volume integral of a function $f(x)$ over an ellipsoid of the described type is equal to zero if the function is odd with respect to the chosen hyperplane. If the hyperplane is given by $x_n = 0$, we thus assume the function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ to be even with respect to x_n : $f(x', x_n) = f(x', -x_n)$ where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Given a Radon-type transform, one is usually interested, among others, in the following questions: uniqueness of reconstruction, inversion formulas and algorithms, and a stability estimate [54, 55]. These are the issues we address below.

This section is organized as follows. The problem is stated precisely in subsection 2.1. Two inversion formulas are presented in subsections 2.2 and 2.3. An analog of the Fourier slice theorem is obtained in subsection 2.2 by taking the Fourier transform with respect to the center and a radial Fourier transform with respect to the half distance between two foci. This theorem plays a critical role in getting a stability estimate and necessary range conditions. The formula discussed in subsection 2.3 is obtained by taking a Fourier type transform and needs less integration than the previous one in subsection 2.2. A stability estimate is handled in subsections 2.4. Subsection 2.5 is devoted to uniqueness for a local data problem. In subsection 2.6, we provide the result of a 2-dimensional numerical simulation.

2.1 Formulation of the problem

We consider all solid ellipsoids of rotation in \mathbb{R}^n with a fixed eccentricity $1/\lambda$, where $\lambda > 1$ and foci located in the hyperplane $x_n = 0$. We will identify this hyperplane with \mathbb{R}^{n-1} . The set of such ellipsoids depends upon $2n - 2$ parameters,

which is $n - 2$ too many. To reduce the overdeterminacy, we require that the focal axis is parallel to a given line, for instance, the x_1 coordinate axis.

Let $u \in \mathbb{R}^{n-1}$ be the center of such an ellipsoid and let $t > 0$ be the half of the focal distance. We denote this ellipsoid by $E_{u,t}$. Then, the foci are

$$c_1 = (u_1 + t, u_2, \dots, u_{n-1}, 0) \text{ and } c_2 = (u_1 - t, u_2, \dots, u_{n-1}, 0)$$

and the points $x \in E_{u,t}$ are described as follows:

$$\frac{(x_1 - u_1)^2}{\lambda^2} + \frac{(x_2 - u_2)^2}{\lambda^2 - 1} + \dots + \frac{x_n^2}{\lambda^2 - 1} \leq t^2.$$

To shorten the formulas, we are going to use the following notation:

$$\nu := \sqrt{\lambda^2 - 1}.$$

The elliptical Radon transform R_E maps a locally integrable function $f(x)$ into its integrals over the solid ellipsoids $E_{u,t}$ for all $u \in \mathbb{R}^{n-1}$ and $t > 0$:

$$R_E f(u, t) = \int_{E_{u,t}} f(x) dx.$$

Our goals are to reconstruct f from $R_E f$ and to study properties of this transform.

2.2 Inversion of the elliptical Radon transform

In this subsection, we assume $f \in C_c^\infty(\mathbb{R}^n)$, i.e., f is a smooth function with a compact support. Here is our strategy. First of all, we change the ellipsoid volume integral to the ellipsoid surface integral, differentiating with respect to t . Second, we take the Fourier transform of this derivative of $R_E f$ with respect to u . Next, taking

a radial Fourier transform with respect to t , we obtain an analog of the Fourier slice theorem.

We introduce a back projection operator R_E^* for $g(u, t) \in C_c^\infty(\mathbb{R}^{n-1} \times \mathbb{R}_+)$ as

$$R_E^*g(x) = \int_{\mathbb{R}^{n-1}} g\left(u, \sqrt{\frac{|u_1 - x_1|^2}{\lambda^2} + \frac{|\tilde{u} - \tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}}\right) du. \quad (2.1)$$

In fact, $R_E^*g(x)$ is the dual transform not to $R_E f(u, t)$, but rather to $\frac{\partial}{\partial t} R_E f(u, t)$, i.e.,

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial t} R_E f(u, t) g(u, t) du dt = C(\lambda) \int_{\mathbb{R}^n} f(x) R_E^*g(x) dx. \quad (2.2)$$

Let χ_S denote the characteristic function of a set $S \subset \mathbb{R}^n$:

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Then the elliptical Radon transform can be written as

$$\begin{aligned} R_E f(u, t) &= \int_{\mathbb{R}^n} \chi_{E_{u,t}} f(x) dx = C(\lambda) \int_{\mathbb{R}^n} \chi_{|x| < t} f(\lambda x_1 + u_1, \nu \tilde{x} + \tilde{u}, \nu x_n) dx \\ &= C(\lambda) \int_0^t r^{n-1} \int_{S^{n-1}} f(\lambda r y_1 + u_1, \nu r \tilde{y} + \tilde{u}, \nu r y_n) d\sigma(y) dr, \end{aligned} \quad (2.3)$$

where $u = (u_1, \tilde{u}) \in \mathbb{R}^{n-1}$, $x = (x_1, \tilde{x}, x_n) \in \mathbb{R}^n$, $C(\lambda) = \lambda \nu^{n-1}$ is the Jacobian factor, and $\sigma(y)$ is the surface measure on S^{n-1} .

Formula (2.3) can be simplified by differentiation with respect to t and division

by t^{n-1} , which yield

$$\begin{aligned} \frac{1}{t^{n-1}} \frac{\partial}{\partial t} R_E f(u, t) &= C(\lambda) \int_{|y|=1} f(\lambda t y_1 + u_1, \nu t \tilde{y} + \tilde{u}, t \nu y_n) d\sigma(y) \\ &= 2C(\lambda) \int_{|y'|\leq 1} f(u + (t\lambda y_1, t\nu \tilde{y}), t\nu \sqrt{1 - |y'|^2}) \frac{dy'}{\sqrt{1 - |y'|^2}}, \end{aligned} \quad (2.4)$$

where $y' = (y_1, \tilde{y}) \in \mathbb{R}^{n-1}$.

It is easy to check that R_E is invariant under the shift with respect to the first $n - 1$ variables. That is, if $f_a(x) := f(x' + a, x_n)$ for $x = (x', x_n) \in \mathbb{R}^n$ and $a \in \mathbb{R}^{n-1}$, we have

$$(R_E f_a)(u, t) = (R_E f)(u + a, t).$$

Thus, application of the $(n - 1)$ -dimensional Fourier transform with respect to the center u seems reasonable. Doing this and changing the variable $y' \in \mathbb{R}^{n-1}$ to the polar coordinates $(\theta, s) \in S^{n-1} \times [0, \infty)$, we get

$$\frac{1}{t^{n-1}} \frac{\partial}{\partial t} \widehat{R_E f}(\xi', t) = 2C(\lambda) \int_0^1 \frac{s^{n-2}}{\sqrt{1 - s^2}} \hat{f}(\xi', t\nu \sqrt{1 - s^2}) \int_{S^{n-2}} e^{its(\lambda\theta_1, \nu\tilde{\theta}) \cdot \xi'} d\theta ds,$$

where \hat{f} and $\widehat{R_E f}$ are the Fourier transforms of f and $R_E f$ with respect to the first $n - 1$ coordinates x' of x and u of (u, t) , respectively, and $\theta = (\theta_1, \tilde{\theta}) \in S^{n-2}$.

To compute the inner integral, we use the identity [7, 21]

$$\int_{S^{n-1}} e^{i\xi \cdot \theta} d\theta = (2\pi)^{n/2} |\xi|^{(2-n)/2} J_{(n-2)/2}(|\xi|). \quad (2.5)$$

We thus get

$$\begin{aligned} & \frac{1}{t^{n-1}} \frac{\partial}{\partial t} \widehat{R_E f} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, t \right) \\ &= \omega_n \int_0^1 \frac{s^{n-2}}{\sqrt{1-s^2}} \hat{f} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, t\nu\sqrt{1-s^2} \right) (ts|\xi'|)^{(3-n)/2} J_{(n-3)/2}(ts|\xi'|) ds, \end{aligned}$$

where $\omega_n = 2(2\pi)^{(n-1)/2} C(\lambda)$.

This enables us to get an analog of the Fourier slice Theorem.

Theorem 2.2.1. *For a function $f \in C_c^\infty(\mathbb{R}^n)$ that is even with respect to x_n , the following formula holds:*

$$\hat{f}(\xi) = \frac{|(\lambda\xi_1, \nu\tilde{\xi}, \nu\xi_n)|^{n-2} |\nu\xi_n|}{2^{n+1}\pi^n C(\lambda)^2} \mathcal{F} \left(R_E^* \frac{1}{t^{n-1}} \frac{\partial}{\partial t} R_E f \right) (\xi), \quad (2.6)$$

where $\mathcal{F}f$ is the n -dimensional Fourier transform of f .

Proof. Let us denote the radial Fourier transform by $F_n f(\rho)$, i.e.,

$$F_n f(\rho) := \rho^{1-n/2} \int_0^\infty t^{n/2} J_{(n-2)/2}(t\rho) f(t) dt. \quad (2.7)$$

We recall that if f is a radial function on \mathbb{R}^n , then the Fourier transform \hat{f} of f is also radial and $\hat{f} = (2\pi)^{n/2} F_n f_0$ where $f_0(|x|) = f(x)$ (cf. [68]). Taking this transform of $\frac{1}{t^{n-1}} \frac{\partial}{\partial t} \widehat{R_E f}$ as a function of t , we have for $\xi = (\xi_1, \tilde{\xi}) = \xi' \in \mathbb{R}^{n-1}$,

$$\begin{aligned} & F_n \left(\frac{1}{t^{n-1}} \frac{\partial}{\partial t} \widehat{R_E f} \right) \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \rho \right) \\ &= \omega_n \rho^{1-n/2} \int_0^\infty \int_0^1 t^{\frac{n}{2}} J_{\frac{n-2}{2}}(t\rho) (s|\xi'|)^{\frac{3-n}{2}} J_{\frac{n-3}{2}}(ts|\xi'|) \hat{f} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, t\nu\sqrt{1-s^2} \right) \frac{s^{n-2} ds dt}{\sqrt{1-s^2}}. \end{aligned} \quad (2.8)$$

It is known [20, p. 59 (18) vol.2 or for $n = 2$, p.55 (35) vol.1] that for $a > 0, \beta > 0$, and, $\mu > \nu > -1$,

$$\begin{aligned} & \int_0^\infty x^{\nu+1/2} (x^2 + \beta^2)^{-1/2\mu} J_\mu(a(x^2 + \beta^2)^{1/2}) J_\nu(xy) (xy)^{1/2} dx \\ &= \begin{cases} a^{-\mu} y^{\nu+1/2} \beta^{-\mu+\nu+1} (a^2 - y^2)^{1/2\mu-1/2\nu-1/2} J_{\mu-\nu-1}(\beta(a^2 - y^2)^{1/2}) & \text{if } 0 < y < a, \\ 0 & \text{if } a < y < \infty. \end{cases} \end{aligned} \quad (2.9)$$

To use the above identity, we make the change of variables $(s, t) \rightarrow (x, \beta)$, where $t = \sqrt{x^2 + \beta^2}$ and $s = x/\sqrt{x^2 + \beta^2}$ in (2.8), which gives

$$\begin{aligned} & F_n \left(\frac{1}{t^{n-1}} \frac{\partial}{\partial t} \widehat{R_E f} \right) \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \rho \right) \\ &= \omega_n \rho^{\frac{2-n}{2}} |\xi'|^{\frac{3-n}{2}} \int_0^\infty \int_0^\infty |x| J_{\frac{n-2}{2}}(\rho(x^2 + \beta^2)^{\frac{1}{2}}) (x^2 + \beta^2)^{-\frac{n-2}{4}} J_{\frac{n-3}{2}}(x|\xi'|) \hat{f} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \nu\beta \right) dx d\beta \\ &= \begin{cases} C(\lambda) \frac{2^{n/2+1} \pi^{n/2} \rho^{2-n}}{\sqrt{\rho^2 - |\xi'|^2}} \int_0^\infty \hat{f} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \beta\nu \right) \cos(\beta\sqrt{\rho^2 - |\xi'|^2}) d\beta & \text{if } |\xi'| < \rho, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.10)$$

Substituting $\rho = |\xi|$ yields

$$F_n \left(\frac{1}{t^{n-1}} \frac{\partial}{\partial t} \widehat{R_E f} \right) \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right) = C(\lambda) \frac{2^{n/2+1} \pi^{n/2} |\xi|^{2-n}}{|\xi_n|} \int_0^\infty \hat{f} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \beta\nu \right) \cos(\xi_n \beta) d\beta.$$

Since f is even in x_n , the last integral is the Fourier transform of f with respect to $x - n$, so we get

$$F_n \left(\frac{1}{t^{n-1}} \frac{\partial}{\partial t} \widehat{R_E f} \right) \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right) = \frac{2^{n/2+1} \pi^{n/2} |\xi|^{2-n}}{|\xi_n|} \lambda \nu^{n-2} \hat{f} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right). \quad (2.11)$$

Taking the Fourier transform of R_E^*g with respect to x yields

$$\begin{aligned}
\widehat{R_E^*g} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right) &= \int_{\mathbb{R}^n} e^{-ix \cdot \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right)} R_E^*g(x) dx \\
&= \int_{\mathbb{R}^n} e^{-ix \cdot \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right)} \int_{\mathbb{R}^{n-1}} g \left(u, \sqrt{\frac{|u_1 - x_1|^2}{\lambda^2} + \frac{|\tilde{u} - \tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}} \right) du dx \\
&= \int_{\mathbb{R}^{n-1}} e^{-iu \cdot \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu} \right)} \int_{\mathbb{R}^n} e^{-i(x' - u, x_n) \cdot \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right)} g \left(u, \sqrt{\frac{|u_1 - x_1|^2}{\lambda^2} + \frac{|\tilde{u} - \tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}} \right) dx du \\
&= C(\lambda) \int_{\mathbb{R}^{n-1}} e^{-iu \cdot \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu} \right)} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} g(u, |x|) dx du \\
&= (2\pi)^{n/2} C(\lambda) \int_{\mathbb{R}^{n-1}} e^{-iu \cdot \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu} \right)} (F_n g(u, \cdot))(|\xi|) du \\
&= (2\pi)^{n/2} C(\lambda) F_n \widehat{g} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right),
\end{aligned} \tag{2.12}$$

where $x = (x', x_n) = (x_1, \tilde{x}, x_n) \in \mathbb{R}^n$, $u = (u_1, \tilde{x}) \in \mathbb{R}^{n-1}$ and $\xi = (\xi', \xi_n) = (\xi_1, \tilde{\xi}, \xi_n) \in \mathbb{R}^n$. Combining (2.11) and (2.12), we get (2.6). \square

Remark 2.2.1. *Theorem 2.2.1 leads naturally to a Fourier type inversion formula for even functions, if one supplements (2.6) with the inverse Fourier transform.*

One can also obtain a useful relation with convolution.

Proposition 2.2.1. *Let $\phi \in C_c^\infty(\mathbb{R}^{n-1} \times [0, \infty))$ and $f \in C_c^\infty(\mathbb{R}^n)$ be even in x_n . If $\psi = R_E^* \phi$ and $g = \frac{1}{t^{n-1}} \frac{\partial}{\partial t} R_E f$. Then we have*

$$g * \phi = \frac{(2\pi)^{n/2}}{C(\lambda)t^{n-1}} \frac{\partial}{\partial t} R_E(f * \psi),$$

where

$$g * \phi(u, |\omega|) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} g(u - u', |\omega - \omega'|) \phi(u', |\omega'|) du' d\omega'.$$

Proof. Note that since $(2\pi)^{n/2}F_nf_0 = \hat{f}$ for a radial function f on \mathbb{R}^n and $f_0(|x|) = f(x)$, we get $F_n(f * g) = (2\pi)^{n/2}F_nfF_ng$. Taking the Fourier transform of $g * \phi$ with respect to u and F_n with respect to t , we get

$$\begin{aligned} \widehat{F_ng * \phi} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right) &= (2\pi)^{n/2}F_n\hat{g} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right) F_n\hat{\phi} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right) \\ &= \frac{(2\pi)^n}{C(\lambda)}F_n\hat{g} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right) \hat{\psi} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right). \end{aligned}$$

In the last line we used (2.12). Equation (2.11) implies

$$\begin{aligned} \widehat{F_ng * \phi} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right) &= \frac{2^{3n/2+1}\pi^{3n/2}|\xi|^{2-n}\nu^{n-2}\lambda}{C(\lambda)|\xi_n|} \hat{f} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right) \hat{\psi} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right) \\ &= \frac{2^{3n/2+1}\pi^{3n/2}|\xi|^{2-n}\nu^{n-2}\lambda}{C(\lambda)|\xi_n|} \widehat{f * \phi} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right) \\ &= \frac{(2\pi)^n}{C(\lambda)}F_n \left(\frac{1}{t^{n-1}} \frac{\partial}{\partial t} R_E \widehat{f * \phi} \right) \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right), \end{aligned}$$

which proves our assertion. □

2.3 A different inversion method

In this subsection, we provide a different inversion formula for the elliptical Radon transform. To obtain this formula, we start to take a transform, which is like the Fourier transform, but with kernel $e^{i\omega t^2}$ instead of $e^{i\omega t}$, of the derivative of $R_E f$ in t . To get the Fourier transform of f from this transform, we change variables.

We start from (2.4). Let us define

$$\begin{aligned}
G(u, w) &:= \int_0^\infty \frac{\partial}{\partial t} R_E f(u, t) e^{iwt^2} dt \\
&= C(\lambda) \int_0^\infty t^{n-1} \int_{|y|=1} f(\lambda ty_1 + u_1, \nu t\tilde{y} + \tilde{u}, \nu y_n) e^{iwt^2} d\sigma(y) dt \\
&= C(\lambda) \int_{\mathbb{R}^{n-1}} f(\lambda y_1 + u_1, \nu \tilde{y} + \tilde{u}, \nu y_n) e^{iw|y|^2} dy,
\end{aligned}$$

where in the last equality we switched from polar to Cartesian coordinates.

Theorem 2.3.1. *Let $f \in C_c^\infty(\mathbb{R}^n)$ with $f(x', x_n) = f(x', -x_n)$. Then we have*

$$f(x) = \frac{x_n}{(2\pi)^n C(\lambda)^2} \int_{\mathbb{R}^n} e^{-i\frac{|\alpha|^2}{4\gamma}} e^{i\alpha \cdot \left(\frac{x_1}{\lambda}, \frac{\tilde{x}}{\nu}\right)} e^{-i\gamma \left(\frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}\right)} G\left(\frac{\alpha_1 \lambda}{2\gamma}, \frac{\alpha' \nu}{2\gamma}, \gamma\right) d\alpha d\gamma,$$

for $x_n > 0$, where $C(\lambda) = \lambda \nu^{n-1}$, as before.

Proof. Making the change of variables $x_1 = \lambda ty_1 + u_1$, $\tilde{x} = \nu t\tilde{y} + \tilde{u}$, $x_n = \nu y_n$, we get

$$\begin{aligned}
G(u, w) &= \int_{\mathbb{R}^n} f(x) e^{iw \left(\frac{(x_1 - u)^2}{\lambda^2} + \frac{|\tilde{x} - \tilde{u}|^2}{\nu^2} + \frac{x_n^2}{\nu^2} \right)} dx \\
&= e^{iw \frac{u^2}{\lambda^2}} e^{iw \frac{|\tilde{u}|^2}{\nu^2}} \int_{\mathbb{R}^n} f(x) e^{iw \left(\frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2} \right)} e^{-2iwu_1 \frac{x_1}{\lambda^2}} e^{-2iw \frac{\tilde{u} \cdot \tilde{x}}{\nu^2}} dx,
\end{aligned}$$

where $x = (x_1, \tilde{x}, x_n)$ and $u = (u_1, \tilde{u}) \in \mathbb{R}^{n-1}$. Next, make the change of variables

$$\mathbf{x}_1 = \frac{x_1}{\lambda}, \quad \tilde{\mathbf{x}} = \frac{\tilde{x}}{\nu}, \quad \text{and} \quad r = \frac{x^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2},$$

so that

$$x_1 = \mathbf{x}_1 \lambda, \quad \tilde{x} = \tilde{\mathbf{x}} \nu, \quad \text{and} \quad x_n = \nu \sqrt{r - \mathbf{x}_1^2 - |\tilde{\mathbf{x}}|^2}.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \nu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-x_1\nu}{2\sqrt{r-x_1^2-\tilde{x}^2}} & \frac{-\tilde{x}\nu}{2\sqrt{r-x_1^2-|\tilde{x}|^2}} & \cdots & \frac{\nu}{2\sqrt{r-x_1^2-|\tilde{x}|^2}} \end{vmatrix} = \frac{C(\lambda)}{2\sqrt{r-x_1^2-|\tilde{x}|^2}}$$

so that

$$dx = \frac{C(\lambda)}{2\sqrt{r-x_1^2-|\tilde{x}|^2}} dx_1 d\tilde{x} dr.$$

Let the function $k(x, \tilde{x}, r)$ be defined by

$$k(x, \tilde{x}, r) = \begin{cases} \frac{f(\lambda x_1, \nu \tilde{x}, \nu \sqrt{r-x_1^2-|\tilde{x}|^2})}{2\sqrt{r-x_1^2-|\tilde{x}|^2}} & 0 < |x_1|^2 + |\tilde{x}|^2 < r, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is even in x_n , it is sufficient to consider the positive root of $\sqrt{r-x_1^2-|\tilde{x}|^2}$.

Then we can rewrite $G(u, w)$ as

$$\begin{aligned} G(u, w) &= C(\lambda) e^{iw\frac{u_1^2}{\lambda^2}} e^{iw\frac{|\tilde{u}|^2}{\nu^2}} \int_{\mathbb{R}^n} k(x_1, \tilde{x}, r) e^{iwr} e^{-2i\frac{wx_1u}{\lambda}} e^{-2i\frac{w\tilde{x}\cdot\tilde{u}}{\nu}} dx_1 d\tilde{x} dr \\ &= C(\lambda) e^{iw\frac{u_1^2}{\lambda^2}} e^{iw\frac{|\tilde{u}|^2}{\nu^2}} K\left(2\frac{wu_1}{\lambda}, 2\frac{w\tilde{u}}{\nu}, -w\right), \end{aligned}$$

where for $\alpha = (\alpha_1, \alpha') \in \mathbb{R} \times \mathbb{R}^{n-2}$,

$$\begin{aligned} K(\alpha, \gamma) &= \int_{\mathbb{R}^n} e^{-i\alpha\cdot(x_1, \tilde{x})} e^{-i\gamma r} k(x_1, \tilde{x}, r) dx_1 d\tilde{x} dr \\ &= \frac{1}{C(\lambda)} e^{i\frac{|\alpha|^2}{4\gamma}} G\left(\frac{-\alpha_1\lambda}{2\gamma}, \frac{-\alpha'\nu}{2\gamma}, -\gamma\right). \end{aligned}$$

Since $k(x, \tilde{x}, r)$ is

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\alpha_1 x_1} e^{i\alpha' \cdot \tilde{x}} e^{i\gamma r} K(\alpha, \gamma) d\alpha d\gamma,$$

we get for $x_n > 0$,

$$\begin{aligned} f(x) &= \frac{x_n}{C(\lambda)} k\left(\frac{x_1}{\lambda}, \frac{\tilde{x}}{\nu}, \frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}\right) \\ &= \frac{x_n}{(2\pi)^n C(\lambda)} \int_{\mathbb{R}^n} e^{i\alpha_1 \frac{x_1}{\lambda}} e^{i\alpha' \cdot \frac{\tilde{x}}{\nu}} e^{i\gamma \left(\frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}\right)} K(\alpha, \gamma) d\alpha d\gamma \\ &= \frac{x_n}{(2\pi)^n C(\lambda)^2} \int_{\mathbb{R}^n} e^{-i\frac{|\alpha|^2}{4\gamma}} e^{i\alpha \cdot \left(\frac{x_1}{\lambda}, \frac{\tilde{x}}{\nu}\right)} e^{-i\gamma \left(\frac{x_1^2}{\lambda^2} + \frac{|\tilde{x}|^2}{\nu^2} + \frac{x_n^2}{\nu^2}\right)} G\left(\frac{\alpha_1 \lambda}{2\gamma}, \frac{\alpha' \nu}{2\gamma}, \gamma\right) d\alpha d\gamma. \end{aligned} \tag{2.13}$$

□

2.4 A stability estimate

In this subsection, we obtain a stability estimate for the elliptical Radon transform. Let $\mathcal{H}^\gamma(\mathbb{R}^n)$ be the regular Sobolev space with a norm

$$\|f\|_\gamma^2 := \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^\gamma d\xi.$$

Let us define $\mathcal{H}_e^\gamma(\mathbb{R}^n) = \{f \in \mathcal{H}^\gamma(\mathbb{R}^n) : f \text{ is even in } x_n\}$ and let $L_{n-1}^2(\mathbb{R}^{n-1} \times [0, \infty))$ be the set of a function g on $\mathbb{R}^{n-1} \times [0, \infty)$ with

$$\|g\|^2 := \int_{\mathbb{R}^{n-1}} \int_0^\infty |g(u, t)|^2 t^{n-1} dt du < \infty.$$

Then $L_{n-1}^2(\mathbb{R}^{n-1} \times [0, \infty))$ is the Hilbert space. Also, by the Plancherel formula, we have $\|g\| = (2\pi)^{2n-1} \|\tilde{g}\|$, where

$$\tilde{g}(\xi, |\zeta|) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} g(u, |w|) e^{-i(u,w) \cdot (\xi, \zeta)} dp dw.$$

Let $\mathcal{H}^\gamma(\mathbb{R}^{n-1} \times [0, \infty))$ be the set of a function $g \in L_{n-1}^2(\mathbb{R}^{n-1} \times [0, \infty))$ with $\|g\|_\gamma < \infty$, where

$$\|g\|_\gamma^2 := \int_{\mathbb{R}^{n-1}} \int_0^\infty |\tilde{g}(\xi', \eta)|^2 (1 + |\xi'|^2 + |\eta|^2)^\gamma \eta^{n-1} d\eta d\xi'.$$

Theorem 2.4.1. *For $\gamma \geq 0$, there is a constant C_n such that $f \in \mathcal{H}_e^\gamma(\mathbb{R}^n)$,*

$$\|f\|_\gamma \leq C_n \|t^{1-n} \partial_t R_E f\|_{\gamma+(n-1)/2}.$$

Proof. Let $g = t^{1-n} \partial_t R_E f$. Note that from (2.12), we have

$$\widehat{R_E^* g} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, \frac{\xi_n}{\nu} \right) = C(\lambda) \int_{\mathbb{R}^{n-1}} e^{-iu \cdot \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu} \right)} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} g(u, |x|) dx du = C(\lambda) \tilde{g} \left(\frac{\xi_1}{\lambda}, \frac{\tilde{\xi}}{\nu}, |\xi| \right). \quad (2.14)$$

Combining this equation and Theorem 2.2.1, we have

$$\hat{f}(\xi) = \frac{|(\lambda \xi_1, \nu \tilde{\xi}, \nu \xi_n)|^{n-2} |\nu \xi_n|}{2^{n+1} \pi^n C(\lambda)} \tilde{g} \left(\xi_1, \tilde{\xi}, |(\lambda \xi_1, \nu \tilde{\xi}, \nu \xi_n)| \right).$$

Hence, we have

$$\begin{aligned}
\|f\|_\gamma^2 &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^\gamma |\hat{f}(\xi)|^2 d\xi \\
&= \frac{1}{2^{2n+2}\pi^{2n}C(\lambda)^2} \int_{\mathbb{R}^n} |(\lambda\xi_1, \nu\tilde{\xi}, \nu\xi_n)|^{2n-4} |\nu\xi_n|^2 (1 + |\xi|^2)^\gamma |\tilde{g}(\xi_1, \tilde{\xi}, |(\lambda\xi_1, \nu\tilde{\xi}, \nu\xi_n)|)|^2 d\xi \\
&\leq C_n \int_{\mathbb{R}^n} |(\lambda\xi_1, \nu\tilde{\xi}, \nu\xi_n)|^{2n-4} |\nu\xi_n|^2 (1 + |(\lambda\xi_1, \nu\tilde{\xi}, \nu\xi_n)|^2)^\gamma |\tilde{g}(\xi_1, \tilde{\xi}, |(\lambda\xi_1, \nu\tilde{\xi}, \nu\xi_n)|)|^2 d\xi \\
&\leq C_n \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} \int_0^\infty (\eta^2 - \lambda^2\xi_1^2 - \nu^2|\tilde{\xi}|^2)^{\frac{1}{2}} \eta^{2n-3} (1 + \eta^2)^\gamma |\tilde{g}(\xi_1, \tilde{\xi}, \eta)|^2 d\eta d\xi_1 d\tilde{\xi}.
\end{aligned}$$

In the last line, we change the variable ξ_n to $\eta = |(\lambda\xi_1, \nu\tilde{\xi}, \nu\xi_n)|$. □

2.5 Uniqueness for the local problem

Theorem 2.2.1 implies that an even function $f \in C_c^\infty(\mathbb{R}^n)$ is uniquely determined by $R_E f$. The question arises if f is uniquely determined by some partial information.

Theorem 2.5.1. *Let $u^0 \in \mathbb{R}^{n-1}$, $\epsilon > 0$, and $T > 0$ be arbitrary. Let $f \in C_c^\infty(\mathbb{R}^n)$ be even in x_n and suppose $g = R_E f$ is equal to zero on the open set*

$$U_{T,\epsilon} = \{(u, t) \in \mathbb{R}^{n-1} \times [0, \infty) : |u - u^0| < \epsilon, 0 \leq t < T\} \text{ (cf. Figure 2.1)}.$$

Then f equals zero on the open set

$$V_T = \left\{ x \in \mathbb{R}^n : \frac{(x_1 - u_1^0)^2}{\lambda^2} + \frac{(\tilde{x} - \tilde{u}^0)^2}{\nu^2} + \frac{x_n^2}{\nu^2} < T^2 \right\}.$$

Here $x = (x_1, \tilde{x}, x_n) \in \mathbb{R}^n$ and $u^0 = (u_1^0, \tilde{u}^0) \in \mathbb{R}^{n-1}$. Also, g is equal to zero on the open cone

$$W_T = \{(u, t) \in \mathbb{R}^{n-1} \times [0, T) : |u - u_0| + t < T\}.$$

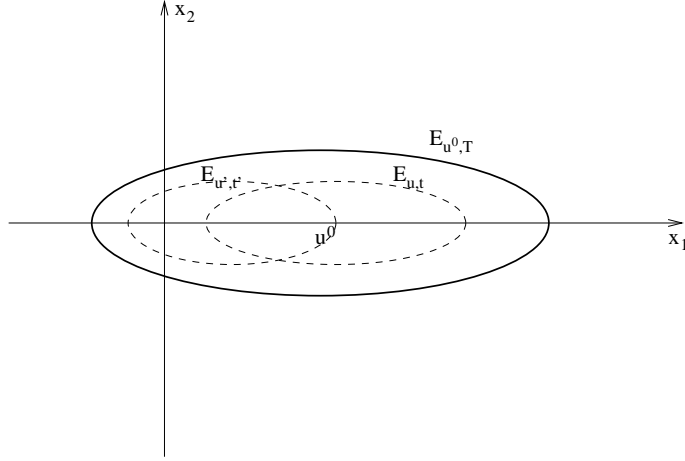


Figure 2.1: Ellipses $E_{u^0, T}$, $E_{u, t}$, $E_{u', t'}$ when $n = 2$

Proof. Without loss of generality, we may assume $u^0 = 0$. Let $f \in C^\infty(\mathbb{R}^n)$. Clearly, g is also differentiable. Differentiating $R_E f(u, t)$ with respect to u_i yields

$$\begin{aligned} \frac{\partial}{\partial u_i} R_E f(u, t) &= C(\lambda) \int \chi_{|x| < t} \frac{\partial}{\partial u_i} f(\lambda x_1 + u_1, \nu \tilde{x} + \tilde{u}, \nu x_n) dx \\ &= C(\lambda) \frac{1}{t} \int_{|x|=t} x_i \frac{\partial}{\partial x_i} f(\lambda x_1 + u_1, \nu \tilde{x} + \tilde{u}, \nu x_n) dx. \end{aligned}$$

Here we used (2.3) and the divergence theorem. Using (2.4), we get

$$\begin{aligned} \frac{\partial}{\partial t} R_E(x_i f)(u, t) &= C(\lambda) \int_{|x|=t} (\nu x_i + u_i) f(\lambda x_1 + u_1, \nu \tilde{x} + \tilde{u}, \nu x_n) d\sigma(x) \\ &= C(\lambda) \left(t\nu \frac{\partial}{\partial u_i} g(u, t) + u_i \frac{\partial}{\partial t} g(u, t) \right). \end{aligned}$$

Let the linear operator D_i be defined by $D_i g(u, t) = C(\lambda)(t\nu \partial_{u_i} g(u, t) + u_i \partial_t g(u, t))$. Then $\frac{\partial}{\partial t} R_E(x_i f)(u, t)$ is $D_i g(u, t)$. By iteration, we obtain $\frac{\partial}{\partial t} R_E(p(x')f) = p(D)g$ where p is an $n - 1$ -variable polynomial. If g is zero in $U_{T, \epsilon}$, then $p(D)g$ is also zero

in $U_{T,\epsilon}$. Then we have for any point $(u, t) \in U_{T,\epsilon}$,

$$\begin{aligned}
\frac{\partial}{\partial t} R_E(p(x')f)(u, t) &= C(\lambda) \int_{|x|=t} p(\lambda x_1 + u_1, \nu \tilde{x} + \tilde{u}) f(\lambda x_1 + u_1, \nu \tilde{x} + \tilde{u}, \nu x_n) d\sigma(x) \\
&= C(\lambda) \int_{|y|<t} p(u + (\lambda y_1, \nu \tilde{y})) f(u + (\lambda y_1, \nu \tilde{y}), \nu \sqrt{t^2 - |y|^2}) \frac{dy}{\sqrt{t^2 - |y|^2}} \\
&= 0.
\end{aligned}$$

For fixed u and t , choose a sequence of polynomials such that $p_i(u + (\lambda y_1, \nu \tilde{y}))$ converge to $f(u + (\lambda y_1, \nu \tilde{y}), \nu \sqrt{t^2 - |y|^2})$ uniformly for $|y| \leq t$ and $y = (y_1, \tilde{y}) \in \mathbb{R}^{n-1}$. It follows that $f = 0$ in V_T and that $g = 0$ in W_T . \square

2.6 Two dimensional numerical implementation

In this subsection, we illustrate our inversion procedure with numerical example. From (2.6), we have the following reconstruction formula:

$$f(x) = \frac{1}{2\pi\lambda^2\nu} \int_{\mathbb{R}^2} |\xi_2| \mathcal{F} \left(R_E^* \frac{1}{t} \frac{\partial}{\partial t} R_E f \right) (\xi) e^{i\xi \cdot x} d\xi.$$

Our algorithm results from the straightforward discretization of the above formula. We set an eccentricity $1/\lambda = 10/11$, i.e., $\lambda = 1.1$. In figure 2.2, there are the $2^8 \times 2^8$ images. The formula (2.4) is approximated by the forward Euler method

$$\frac{\partial}{\partial t} R_E f(u, t) \approx \frac{R_E f(u, t + \Delta t) - R_E f(u, t)}{\Delta t},$$

where Δt is the discretization step. Our phantom function, supported within the square with side length 2, is the sum of characteristic functions of six disks. Since it has to be even in x_2 and there are three characteristic functions of disks with radii 0.2, 0.25 and 0.3 above the x_1 -axis, we also include their reflection below the x_1 -axis.

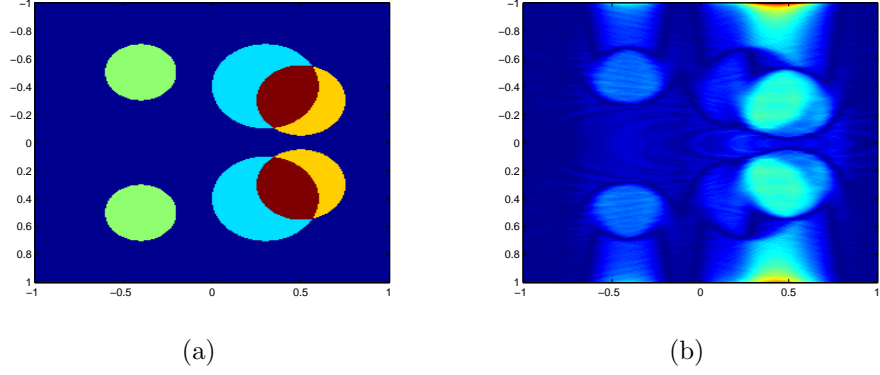


Figure 2.2: Two dimensional numerical implementation: (a) the phantom and (b) reconstruction ($\lambda = 1.1$)

We compute $R_E f(u, t)$ for u ranging from -128 to 128 , and t ranging from 0 to 257 ¹ with the discretization step $\Delta t = 257/N$, where N is 2^{11} as the number of taken samples. These measurements were used to produce the reconstruction in figure 2.2 (b).

¹To implement a back projection (2.1) properly, the range of t should have a little wider than that of u .

3. SINGLE SCATTERING OPTICAL TOMOGRAPHY AND THE V-LINE RADON TRANSFORM*

Optical tomography uses measurements of light transmitted and scattered in the body to produce images of its interior. Under certain reasonable assumptions (e.g. when the optical thickness of the body part is small), one can assume that the light photons scatter at most once inside the body [26, 27, 28]. Hence they travel along piecewise-linear trajectories (**broken rays**) on their paths from the emitters to the receivers. Using angularly focused emitters and receivers in the plane, one can measure the change of light intensity along various broken rays, and then utilize this information to recover the spatially variable coefficients of light absorption and scattering. The latter functions are then used to create images of 2D slices of the body, which are then stacked together to generate a full 3D image of the interior. This technique is called single-scattering optical tomography (SSOT). Mathematically, the image reconstruction problem in SSOT requires inversion of the V-line Radon transform (VRT) integrating a function of two variables along broken rays, which look like V-shaped piecewise linear trajectories (hence the name) [26, 27, 28].

Subsection 3.1 is devoted to definition of V-line Radon transform. Inversion formula is presented in subsection 3.2. Also, we provide some remarks and comments in subsection 3.3.

3.1 V-line Radon transform

Let the function $f(x, y)$ be defined inside the disc $D(0, R)$ of radius R centered at the origin and let $\theta \in (0, \pi/2)$ be a fixed angle. Denote by $BR(\beta, t)$ the broken ray

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that emanates from the point $A(\beta) = (R \cos \beta, R \sin \beta)$ on the boundary of $D(0, R)$, travels the distance $d = R - t$ along the diameter (i.e. normally to the boundary) to point $B(\beta, t)$, then breaks into another ray under the obtuse angle $\pi - \theta$ arriving at point $C(\beta, t)$ (see Figures 3.1, 3.2).

The V-line Radon transform of function f is the integral

$$VRf(\beta, t) = \int_{BR(\beta, t)} f ds, \quad \beta \in [0, 2\pi], \quad t \in [0, R],$$

of $f(x, y)$ along the broken ray $BR(\beta, t)$ with respect to linear measure ds .

In the circular setup of SSOT the point $A(\beta)$ corresponds to the location of the light source, the points $C_j = C(\beta, t_j)$ correspond to the locations of (an array of) receivers, and $B_j = B(\beta, t_j)$ are the scattering points. After making the measurements for all possible angles $\beta \in [0, 2\pi]$ of the emitter of normally incident beams and all scattering distances $t \in [0, R]$, one obtains a two-dimensional family of VRf data. The problem of image reconstruction in SSOT then requires inverting VRf , i.e. finding $f(x, y)$ from the measured data $VRf(\beta, t)$.

3.2 Inversion of the VRT

In a remark in [2] an approach to inverting the VRT was mentioned that utilizes the fact that our transform integrates along a rotationally invariant family of broken rays. Such strategy has been successfully used in the past for inversion of other generalized Radon transforms (e.g. see [5, 54, 55]). The main idea of this approach is based on the fact that the rotational invariance should allow one to diagonalize the VRT when passing to the “basis of exponentials”. In other words, when the unknown function f is presented in polar coordinates and then expanded to a Fourier series with respect to the polar angle, its n -th Fourier coefficient should depend only on the

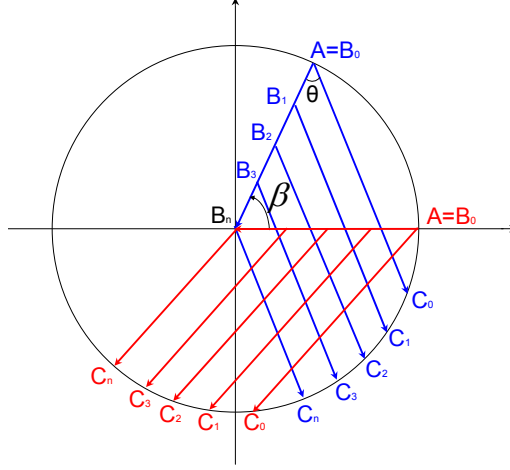


Figure 3.1: A sketch of the SSOT setup in circular geometry. $A(\beta)$ corresponds to the location of the light source, the points $C_j = C(\beta, t_j)$ correspond to the locations of (an array of) receivers, and $B_j = B(\beta, t_j)$ are the scattering points, where t_j is the distance from the breaking point to the origin. Our method uses data only from the rays that scatter before they reach the center of the disc.

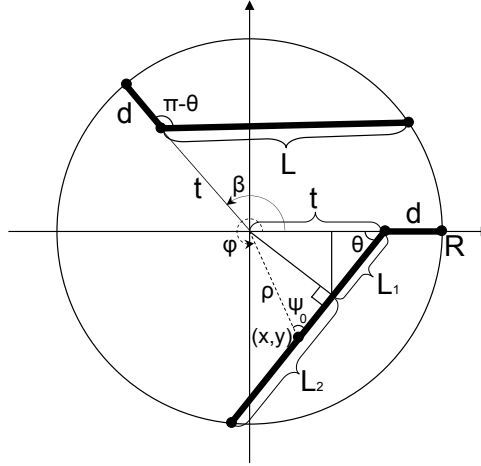


Figure 3.2: A sketch of the domain and the notations. Here R is the fixed radius of the circular trajectory of the emitter and receivers, θ is the fixed scattering angle, β is the polar angle of the emitter, d is the distance traveled by the ray before breaking (scattering), $t = R - d$, and the broken-rays are parameterized using the ordered pair (β, t) .

n -th Fourier coefficient of VRf expanded into Fourier series with respect to its own angular variable. As a result, the problem should reduce to an Abel-type integral equation with a special function kernel, which can be either solved explicitly, or through an iterative procedure. In this subsection we present a detailed exposition of this strategy and state the main results.

For brevity let us denote $g(\beta, t) := VRf(\beta, t)$, and let $f(\phi, \rho)$ be the image function in polar coordinates. Then the Fourier series of $f(\phi, \rho)$ and $g(\beta, t)$ with respect to their angular variables can be written as follows

$$f(\phi, \rho) = \sum_{n=-\infty}^{\infty} f_n(\rho) e^{in\phi}, \quad g(\beta, t) = \sum_{n=-\infty}^{\infty} g_n(t) e^{in\beta},$$

where the Fourier coefficients are given by

$$f_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi, \rho) e^{-in\phi} d\phi, \quad g_n(t) = \frac{1}{2\pi} \int_0^{2\pi} g(\beta, t) e^{-in\beta} d\beta.$$

Using the rotation invariance of VRf (see Figure 3.2), we get

$$g(\beta, t) = \int_{BR(\beta, t)} f(\phi, \rho) ds = \int_{BR(0, t)} f(\phi + \beta, \rho) ds = \sum_{n=-\infty}^{\infty} \int_{BR(0, t)} f_n(\rho) e^{in(\phi + \beta)} ds.$$

Hence, we obtain the following relation between the Fourier coefficients of two functions:

$$g_n(t) = \int_{BR(0, t)} f_n(\rho) e^{in\phi} ds = \int_t^R f_n(\rho) d\rho + \int_L f_n(\rho) e^{in\phi} dl \quad (3.1)$$

where dl is the length measure along the line interval $L := \{(x, y) : y = (x - t) \tan \theta, y < 0, x^2 + y^2 < R^2\}$.

Theorem 3.2.1. *Let f be supported in the disk $D(0, R)$. Then we have*

$$\mathcal{M}f_n(s) = \frac{\mathcal{M}g_n(s-1)}{1/(s-1) + \mathcal{M}h_n(s-1)}, \quad \Re(s) > 1 \quad (3.2)$$

where $\mathcal{M}F$ denotes the Mellin transform of function F and the function h_n is defined piecewise as

$$h_n(t) = (-1)^{n+1} e^{in\psi(t)} \frac{1 + t \cos[\psi(t)] + t^2 \sin[\psi(t)] \frac{\sin \theta}{\sqrt{1-t^2 \sin^2 \theta}}}{\sqrt{1+t^2+2t \cos(\psi(t))}}$$

$$+ e^{in[2\theta-\psi(t)]} \frac{1 - t \cos[2\theta - \psi(t)] + t^2 \sin[2\theta - \psi(t)] \frac{\sin \theta}{\sqrt{1-t^2 \sin^2 \theta}}}{\sqrt{1+t^2-2t \cos[2\theta - \psi(t)]}}, \quad \text{if } 1 < t < \frac{1}{\sin \theta},$$

$$h_n(t) = (-1)^{n+1} e^{in\psi(t)} \frac{1 + t \cos[\psi(t)] + t^2 \sin[\psi(t)] \frac{\sin \theta}{\sqrt{1-t^2 \sin^2 \theta}}}{\sqrt{1+t^2+2t \cos[\psi(t)]}}, \quad \text{if } 0 < t \leq 1,$$

and

$$h_n(t) \equiv 0, \quad \text{for all } t > \frac{1}{\sin \theta}.$$

Here $\psi(t) = \arcsin(t \sin \theta) + \theta$.

Proof. Consider two subintervals L_1 and L_2 of interval L (see Figure 3.2). $L_1 := \{(x, y) : y = (x - t) \tan \theta, \quad t \sin^2 \theta \leq x \leq t\}$ and $L_2 := \{(x, y) : y = (x - t) \tan \theta, \quad -\infty < x < t \sin^2 \theta\}$. Let us find explicit formulas of the dependence $\phi(\rho)$ of polar angle from polar radius for a point (x, y) moving along L_1 and L_2 . For that consider the law of sines inside the triangle having vertices at the origin, $(t, 0)$, and (x, y) (see Figure 3.2 for all notations used here and below).

If (x, y) is located in L_1 , then the angle at that point $\psi_0 = \pi - \arcsin\left(\frac{t}{\rho} \sin \theta\right)$ since ψ_0 is obtuse. Thus, on L_1 the polar angle is $\phi = 2\theta - \psi(t/\rho)$.

Similarly, for $(x, y) \in L_2$ we have $\psi_0 = \arcsin\left(\frac{t}{\rho} \sin \theta\right)$ since ψ_0 is acute, and therefore, the polar angle is $\phi = \pi + \psi(t/\rho)$.

In order to find the length measure dl on L , we get the length l between (x, y) and $(t, 0)$ using the law of cosines. Then $l = \sqrt{\rho^2 + t^2 - 2\rho t \cos \phi}$, and

$$dl = \frac{\rho - t \cos \phi + t \rho \frac{d\phi}{d\rho} \sin \phi}{\sqrt{\rho^2 + t^2 - 2\rho t \cos \phi}} d\rho.$$

Hence we get

$$dl = \begin{cases} \frac{\rho - t \cos \left[2\theta - \psi\left(\frac{t}{\rho}\right)\right] + \frac{t^2}{\rho} \sin \left[2\theta - \psi\left(\frac{t}{\rho}\right)\right] \frac{\sin \theta}{\sqrt{1 - \frac{t^2}{\rho^2} \sin^2 \theta}}}{\sqrt{\rho^2 + t^2 - 2\rho t \cos \left[2\theta - \psi\left(\frac{t}{\rho}\right)\right]}} d\rho & \text{if } (x, y) \in L_1 \\ \frac{\rho - t \cos \left[\pi + \psi\left(\frac{t}{\rho}\right)\right] - \frac{t^2}{\rho} \sin \left[\pi + \psi\left(\frac{t}{\rho}\right)\right] \frac{\sin \theta}{\sqrt{1 - \frac{t^2}{\rho^2} \sin^2 \theta}}}{\sqrt{\rho^2 + t^2 - 2\rho t \cos \left[\pi + \psi\left(\frac{t}{\rho}\right)\right]}} d\rho & \text{if } (x, y) \in L_2. \end{cases}$$

Substituting the obtained expressions for ϕ and dl into (3.1), we get

$$\begin{aligned} g_n(t) &= \int_t^R f_n(\rho) d\rho \\ &- \int_t^{t \sin \theta} f_n(\rho) e^{in[2\theta - \psi(\frac{t}{\rho})]} \frac{\rho - t \cos[2\theta - \psi(\frac{t}{\rho})] + \frac{t^2}{\rho} \sin[2\theta - \psi(\frac{t}{\rho})] \frac{\sin \theta}{\sqrt{1 - \frac{t^2}{\rho^2} \sin^2 \theta}}}{\sqrt{\rho^2 + t^2 - 2\rho t \cos[2\theta - \psi(\frac{t}{\rho})]}} d\rho \\ &- \int_{t \sin \theta}^\infty f_n(\rho) e^{in[\pi + \psi(\frac{t}{\rho})]} \frac{\rho - t \cos[\pi + \psi(\frac{t}{\rho})] - \frac{t^2}{\rho} \sin[\pi + \psi(\frac{t}{\rho})] \frac{\sin \theta}{\sqrt{1 - \frac{t^2}{\rho^2} \sin^2 \theta}}}{\sqrt{\rho^2 + t^2 - 2\rho t \cos[\pi + \psi(\frac{t}{\rho})]}} d\rho \end{aligned}$$

Continuing the computation yields

$$\begin{aligned}
g_n(t) &= \int_t^R f_n(\rho) d\rho \\
&+ \int_{t \sin \theta}^t f_n(\rho) e^{in[2\theta - \psi(\frac{t}{\rho})]} \frac{1 - \frac{t}{\rho} \cos[2\theta - \psi(\frac{t}{\rho})] + \frac{t^2}{\rho^2} \sin[2\theta - \psi(\frac{t}{\rho})] \frac{\sin \theta}{\sqrt{1 - \frac{t^2}{\rho^2} \sin^2 \theta}}}{\sqrt{1 + \frac{t^2}{\rho^2} - 2\frac{t}{\rho} \cos[2\theta - \psi(\frac{t}{\rho})]}} d\rho \\
&+ (-1)^{n+1} \int_{t \sin \theta}^{\infty} f_n(\rho) e^{in\psi(\frac{t}{\rho})} \frac{1 + \frac{t}{\rho} \cos[\psi(\frac{t}{\rho})] + \frac{t^2}{\rho^2} \sin[\psi(\frac{t}{\rho})] \frac{\sin \theta}{\sqrt{1 - \frac{t^2}{\rho^2} \sin^2 \theta}}}{\sqrt{1 + \frac{t^2}{\rho^2} + 2\frac{t}{\rho} \cos[\psi(\frac{t}{\rho})]}} d\rho \\
&= \int_t^R f_n(\rho) d\rho + \{[\rho f_n(\rho)] \times h_n\}(t),
\end{aligned} \tag{3.3}$$

Where

$$\{f \times g\}(s) = \int_0^{\infty} f(\rho) g\left(\frac{s}{\rho}\right) \frac{d\rho}{\rho}.$$

Secon term of last line in (3.3) is a convolution for Mellin transform. Also, since f_n, g_n and h_n have compact support, their Mellin transform is well defined. Doing it on g_n , we get

$$\mathcal{M}g_n(s) = \frac{1}{s} \mathcal{M}f_n(s+1) + \mathcal{M}f_n(s+1) \mathcal{M}h_n(s).$$

Here we use some properties of the Mellin transform, namely: $\mathcal{M}[\rho f(\rho)](s) = \mathcal{M}f(s+1)$ and $\mathcal{M}[\int_t^{\infty} f(x) dx](s) = \mathcal{M}f(s+1)/s$, which are easily verified using the fact that $f_n(\rho) = 0$ for $\rho > R$ (see (1.3)). Now a simple computation gives us the claimed formula. \square

Corollary 3.2.1. *Let $f_n(\rho)$ be the n -th Fourier coefficient of the 2nd differentiable*

function f supported in the disk $D(0, R)$. Then for any $t > 1$ we have

$$f_n(\rho) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{t-Ti}^{t+Ti} \rho^{-s} \frac{\mathcal{M}g_n(s-1)}{1/(s-1) + \mathcal{M}h_n(s-1)} ds. \quad (3.4)$$

Proof. For $a > 1$ and $b \in \mathbb{R}$, $|\mathcal{M}f_n(a+bi)|$ is finite because

$$\int_0^\infty r^{a+bi-1} |f_n(r)| dr \leq C \int_0^R r^{a-1} |e^{ib \ln r}| dr$$

where C is the upper bound of $|f_n|$. Thus, $\mathcal{M}f_n(s)$ is analytic on $\{z \in \mathbb{C} : \Re z > 1\}$.

Using integral by part twice, we get

$$\mathcal{M}f_n(s) = \int_0^\infty f''(\rho) \frac{\rho^{s+1}}{s(s+1)} d\rho,$$

which implies $\mathcal{M}f_n(s) = O(s^2)$. Hence $\mathcal{M}f_n(t+si)$ is integrable and we can applying inverse Mellin transform [25, 70]. Applying it to formula (3.2) gives our claimed formula (3.4). \square

3.3 Some general remarks and comments

1. The numerical implementation of the inversion formula described above is an elaborate task in its own right, and we plan to address it in the future.
2. Our inversion formula uses only data from $VRf(\beta, t)$ for $t \in [0, R]$ and $\beta \in [0, 2\pi)$ to recover f whose support is in the whole disk $D(0, R)$. The previously known formula provided in [2] is using twice more data ($t \in [0, 2R]$ and $\beta \in [0, 2\pi)$) and works for a more restrictive class of functions (supported in $D(0, R \sin \theta)$).
3. Despite the fact that we are not using a full set of data in our inversion,

the remaining set is still microlocally complete. In other words our data set includes integrals along broken rays passing through any given point in the disc and (with one of its two parts) normal to any given direction. This means that theoretically we have enough data for a numerically stable reconstruction. For more details on microlocal properties of Radon-type transforms see, for example [60, 80] and the references there.

4. PHOTOACOUSTIC TOMOGRAPHY AND RELATED RADON-TYPE TRANSFORMS

Photoacoustic Tomography (PAT) is the best-known example of a **hybrid imaging method**. It has applications to functional brain imaging of animals, early cancer diagnostics, and imaging of vasculature [32]. In 1880, A.G. Bell discovered the photo-acoustic effect [9]. This effect enabled one to combine advantages of pure optical and ultrasound imaging, providing both high optical contrast and ultrasonic resolution. Nevertheless, PAT has rather low cost.

In PAT, one induces an acoustic pressure wave inside of an object of interest by delivering optical energy [47, 79]. The acoustic wave on a surface is measured outside of the object. Mathematically, in the model we study, the problem boils down to recovering the initial data of the three dimensional wave equation from the values of the solution observed at all times on the surface. This initial pressure field contains diagnostic information.

Various types of detectors have been considered for measuring the acoustic data: point-like detectors, line detectors, planar detectors, cylindrical detectors, and circular detectors. While point-like detectors approximately measure the pressure at a given point, other types of detectors are integrating. In this section, we study line detectors and circular detectors.

4.1 Line detectors

In this subsection, we consider PAT with line detectors. The line detector renders the integral of the pressure along its length. We obtain this integration value at different moments of time. This data is equivalent to measuring the surface integral over the cylinders with central axis corresponding to a detector line and whose radii

are arbitrary.

Various configurations of line detectors were considered in [12, 13, 32, 33]. In this subsection, we deal with two basic geometries: the line detectors are tangent to a cylinder, and the line detectors are located on a plane. We call these **the cylindrical version** and **the planar version**, respectively. Some inversion formulas for the first version were found in [33]. In this subsection, we address other issues of importance in tomography [54, 55]: a support theorem, a stability estimate, and necessary range conditions. We also consider an n -dimensional case of this model. In the planar version, Haltmeier [32] provided a two-step procedure of image reconstruction. In this subsection, we define a cylindrical Radon transform and present an analog of the Fourier slice theorem as well as a stability estimate, and necessary range conditions.

4.1.1 Cylindrical geometry

We explain first the mathematical model arising in PAT with line detectors as introduced in [33]. Let B_R^k be the ball in \mathbb{R}^k centered at the origin with radius $R > 0$. Then $B_R^2 \times \mathbb{R}$ is the cylinder in \mathbb{R}^3 with radius R . For fixed $p \in \mathbb{R}$ and $\theta \in S^1$, let

$$L_C(\theta, p) = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \cdot \theta = R, z = p\}$$

be the line occupied by a detector. Detector lines $L_C(\theta, p)$ are tangent to the cylinder $B_R^2 \times \mathbb{R}$ (see Figure 4.1).

Definition 4.1.1. *The cylindrical Radon transform R_C maps a function $f \in C_c^\infty(B_R^2 \times \mathbb{R})$ to*

$$R_C f(\theta, p, r) = \frac{1}{2\pi r} \iint_{d(L_C(\theta, p), (x, y, z))=r} f(x, y, z) d\varpi,$$

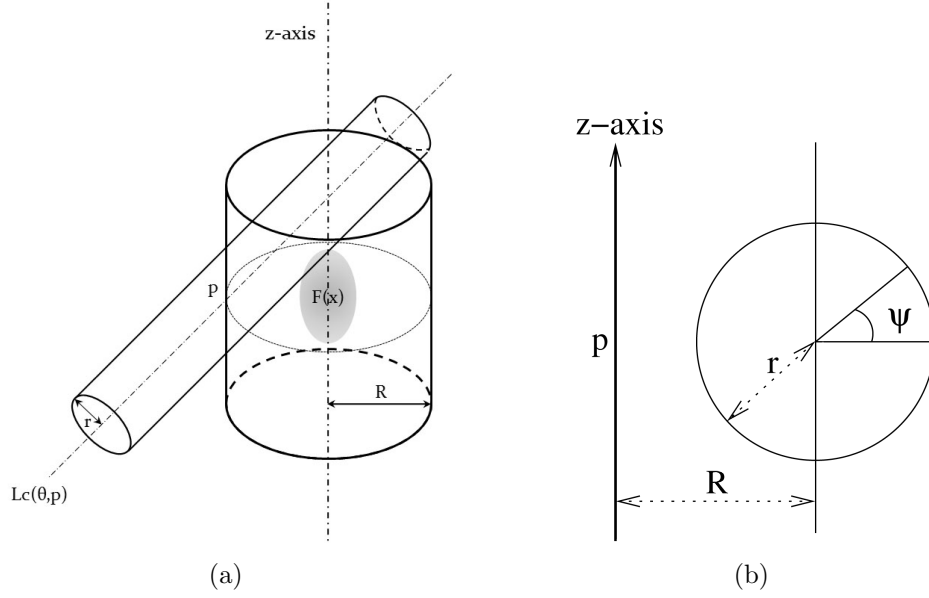


Figure 4.1: (a) the integral area cylinder and the cylinder $B_R^2 \times \mathbb{R}$ in which f has compact support and (b) the restriction to the $\{(t\theta, z) : t \in \mathbb{R}, z \in \mathbb{R}\}$ plane

for $(\theta, p, r) \in S^1 \times \mathbb{R} \times [0, \infty)$. Here $d\varpi$ is the area measure on the cylinder

$$\{(x, y, z) \in \mathbb{R}^3 : d(L_C(\theta, p), (x, y, z)) = r\}$$

and

$$d(L_C(\theta, p), (x, y, z)) := \sqrt{(R - (x, y) \cdot \theta)^2 + (p - z)^2}$$

denotes the Euclidean distance between the line $L_C(\theta, p)$ and the point (x, y, z) .

Remark 4.1.1. When one fixes θ and restricts the cylindrical Radon transform $R_C f$ to the plane $\{(t\theta, z) : t \in \mathbb{R}, z \in \mathbb{R}\}$, $R_C f$ turns into the 2-dimensional circular Radon transform whose centers are located at $(R\theta, p)$ (see Figure 4.1 (b)).

By definition, we have

$$R_C f(\theta, p, r) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} f(t\theta^\perp + (R - r \cos \psi)\theta, p + r \sin \psi) d\psi dt,$$

where p and r are the height and radius, θ is the direction from the z -axis to the central axis of the cylinder, t is a parameter along the central axis of the cylinder, and ψ is the polar angle of the circle that is the intersection of plane $\{(t\theta, z) : t \in \mathbb{R}, z \in \mathbb{R}\}$ and the cylinder.

4.1.1.1 Inversion formulas

We have two integrals in the definition formula of $R_C f$. For fixed θ , the inner integral is a circular Radon transform with centers at the space $\{(R\theta, z) : z \in \mathbb{R}\}$ (see Figure 4.1 (b)). Also, the outer integral can be thought of as the 2-dimensional regular Radon transform for a fixed z variable [33]. We start by applying the inversion of the circular Radon transform for fixed θ .

To obtain inversion formulas, we define the operator R_C^* for $g \in C_c^\infty(S^1 \times \mathbb{R} \times [0, \infty))$ by

$$R_C^* g(\theta, z, \rho) = \int_{\mathbb{R}} g(\theta, p, \sqrt{(z - p)^2 + \rho^2}) dp,$$

for $z \in \mathbb{R}$ and $\rho \in \mathbb{R}$.

We have an analogue of the Fourier slice theorem.

Theorem 4.1.1. *Let $f \in C_c^\infty(B_R^2 \times \mathbb{R})$. If $g = R_C f$, then we have for $(\theta, \sigma, \xi) \in S^1 \times \mathbb{R} \times \mathbb{R}$,*

$$\hat{f}(\sigma\theta, \xi) = \pi^{-1} \widehat{R_C^* g}(\theta, \xi, \sigma) e^{iR\sigma|\sigma|}, \quad (4.1)$$

where \hat{f} is the 3-dimensional Fourier transform of f and $\widehat{R_C^* g}$ is the 2-dimensional Fourier transform of $R_C^* g$ with respect to (z, ρ) .

Remark 4.1.2. We remind readers the Fourier slice theorems for the circular and regular Radon transforms.

When $\mathcal{R}f(\theta, s) = \int_{\theta \cdot x = s} f(x) dx$ for $(\theta, s) \in S^1 \times \mathbb{R}$ is the regular Radon transform, we have $\widehat{\mathcal{R}f}(\theta, \sigma) = \hat{f}(\sigma\theta)$. Also, when $Mf(u, r) = \int_{S^1} f((u, 0) + r\alpha) d\alpha$ for $(u, r) \in \mathbb{R} \times [0, \infty)$ is the circular Radon transform, we have $\hat{f}(\xi) = \widehat{M^*Mf}(\xi)|\xi_2|$, where $M^*g(x, y) = \int_{\mathbb{R}} g(u, \sqrt{(u-x)^2 + y^2}) du$ for a function g on $\mathbb{R} \times [0, \infty)$ [55, 57]. Equation (4.1) can be thought of as the combination of two Fourier slice theorems: for the circular and regular Radon transforms.

Proof of the theorem. Taking the Fourier transform of $R_C f$ with respect to p yields

$$\widehat{R_C f}(\theta, \xi, r) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-1}^1 \hat{f}(t\theta^\perp + (R - r\sqrt{1-s^2})\theta, \xi) e^{irs\xi} \frac{ds}{\sqrt{1-s^2}} dt,$$

where \hat{f} and $\widehat{R_C f}$ are the 1-dimensional Fourier transforms of f and $R_C f$ with respect to z and p , respectively. Taking the Hankel transform of order zero of $\widehat{R_C f}$ with respect to r , we have

$$\begin{aligned} H_0 \widehat{R_C f}(\theta, \xi, \eta) &= \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}} \int_{-1}^1 \hat{f}(t\theta^\perp + (R - r\sqrt{1-s^2})\theta, \xi) e^{irs\xi} \frac{ds}{\sqrt{1-s^2}} dt J_0(r\eta) r dr \\ &= \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}} \int_{-1}^1 \hat{f}(t\theta^\perp + (R - r\sqrt{1-s^2})\theta, \xi) \cos(rs\xi) \frac{ds}{\sqrt{1-s^2}} J_0(r\eta) r dt dr \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \hat{f}(t\theta^\perp + (R - b)\theta, \xi) \cos(\rho\xi) J_0(\eta\sqrt{\rho^2 + b^2}) d\rho db dt, \end{aligned} \tag{4.2}$$

where in the last line, we made a change of variables $(r, s) \rightarrow (\rho, b)$ where $r =$

$\sqrt{\rho^2 + b^2}$ and $s = \rho/\sqrt{\rho^2 + b^2}$. We will use the following identity: for $a, b > 0$

$$\int_0^\infty J_0(a\sqrt{\rho^2 + b^2}) \cos(\rho\xi) d\rho = \begin{cases} \frac{1}{\sqrt{a^2 - \xi^2}} \cos(b\sqrt{a^2 - \xi^2}) & \text{if } 0 < \xi < a, \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$

[20, p.55 (35) vol.1]. Applying this identity to (4.2), we conclude that $H_0\widehat{R_C}f(\theta, \xi, \eta)$ is equal to

$$\begin{cases} \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^\infty \hat{f}(t\theta^\perp + (R-b)\theta, \xi) \frac{1}{\sqrt{\eta^2 - \xi^2}} \cos(b\sqrt{\eta^2 - \xi^2}) db dt & \text{if } 0 < \xi < \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting $\eta = \sqrt{\xi^2 + \sigma^2}$ yields

$$H_0\widehat{R_C}f(\theta, \xi, |(\xi, \sigma)|) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^\infty \hat{f}(t\theta^\perp + (R-b)\theta, \xi) \frac{\cos(b\sigma)}{\sigma} db dt.$$

The inner integral in the right hand side is the Fourier cosine transform with respect to b , so taking the inverse Fourier cosine transform of the above formula, we get

$$\int_{\mathbb{R}} \hat{f}(t\theta^\perp + (R-s)\theta, \xi) dt = 4 \int_0^\infty H_0\widehat{R_C}f(\theta, \xi, |(\xi, \sigma)|) \cos(s\sigma) \sigma d\sigma, \quad (4.4)$$

where \hat{f} is the 1-dimensional Fourier transform of f with respect to the last variable z . For a fixed ξ , one recognizes the Radon transform in the left hand side. We, thus, can apply an inversion of the Radon transform.

Before doing that, we change the right hand side of (4.4) into a term containing the backprojection operator R_C^* . Taking the Fourier transform of R_C^*g on $S^1 \times \mathbb{R}^2$

with respect to the last two variables (z, ρ) yields

$$\begin{aligned}
\widehat{R_C^* g}(\theta, \xi, \sigma) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(z, \rho) \cdot (\xi, \sigma)} R_C^* g(\theta, z, \rho) dz d\rho \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(z, \rho) \cdot (\xi, \sigma)} \int_{\mathbb{R}} g(\theta, p, \sqrt{(z-p)^2 + \rho^2}) dp dz d\rho \\
&= \int_{\mathbb{R}} e^{-i\xi p} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(z-p, \rho) \cdot (\xi, \sigma)} g(\theta, p, \sqrt{(z-p)^2 + \rho^2}) dz d\rho dp \\
&= \int_{\mathbb{R}} e^{-i\xi p} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(z, \rho) \cdot (\xi, \sigma)} g(\theta, p, |(z, \rho)|) dz d\rho dp \\
&= 2\pi \int_{\mathbb{R}} e^{-i\xi \cdot p} H_0 g(\theta, p, |(\xi, \sigma)|) dp \\
&= 2\pi H_0 \hat{g}(\theta, \xi, |(\xi, \sigma)|),
\end{aligned} \tag{4.5}$$

where $\widehat{R_C^* g}$ is the 2-dimensional Fourier transform of g in (z, ρ) . Combining this (4.5) with (4.4), we have for $g = R_C f$,

$$\begin{aligned}
\int_{\mathbb{R}} \hat{f}(t\theta^\perp + s\theta, \xi) dt &= \frac{2}{\pi} \int_0^\infty \widehat{R_C^* g}(\theta, \xi, \sigma) \cos((R-s)\sigma) \sigma d\sigma \\
&= \frac{1}{\pi} \int_{\mathbb{R}} \widehat{R_C^* g}(\theta, \xi, \sigma) e^{i(R-s)\sigma} |\sigma| d\sigma,
\end{aligned} \tag{4.6}$$

since $\widehat{R_C^* g}$ is even in σ by the evenness in ρ of $R_C^* g$. Taking the Fourier transform of (4.6) with respect to s completes the proof. \square

Theorem 4.1.2. *Let $f \in C_c^\infty(B_R^2 \times \mathbb{R})$. If $g = R_C f$, then we have*

$$f(x, y, z) = \frac{1}{4\pi^2} \int_{S^1} I_2^{-2} R_C^* g(\theta, z, \rho) \Big|_{\rho=(x, y) \cdot \theta - R} d\theta, \tag{4.7}$$

where we use the Riesz potential $\widehat{I_2^{-2} h}(\theta, \xi, \sigma) = |\sigma|^2 \hat{h}(\theta, \xi, \sigma)$ for a function $h(\theta, z, \rho)$ on $S^1 \times \mathbb{R}^2$ with its 2-dimensional Fourier transform $\hat{h}(\theta, \xi, \sigma)$ with respect to real

variables.

Proof. Using Theorem 4.1.1, we have

$$\begin{aligned}
f(x, y, z) &= \frac{1}{(2\pi)^3} \int_0^\infty \int_{S^1} \int_{\mathbb{R}} \hat{f}(\sigma\theta, \xi) |\sigma| e^{i(\sigma(x,y)\cdot\theta + z\xi)} d\sigma d\theta d\xi \\
&= \frac{1}{(2\pi)^3 \pi} \int_0^\infty \int_{S^1} \int_{\mathbb{R}} \widehat{R_C^* g}(\theta, \xi, \sigma) e^{iR\sigma} |\sigma|^2 e^{i(\sigma(x,y)\cdot\theta + z\xi)} d\sigma d\theta d\xi \\
&= \frac{1}{(2\pi)^4} \int_{\mathbb{R}} \int_{S^1} \int_{\mathbb{R}} \widehat{R_C^* g}(\theta, \xi, \sigma) e^{iR\sigma} |\sigma|^2 e^{i(\sigma(x,y)\cdot\theta + z\xi)} d\xi d\theta d\sigma.
\end{aligned}$$

□

Remark 4.1.3. *Inversion formula (4.9) is the same as that of [33]. There M. Haltmeier obtained it combining two inversion formulas for the circular Radon transform and the Radon transform. We obtain it instead through an analog of the Fourier slice theorem.*

The equation (4.6) hints that it is natural to try to use another inversion of the Radon transform, the one using circular harmonics. Let $f(t, \varphi, z)$ be the image function in cylindrical coordinates. Then the Fourier series of $f(t, \varphi, z)$ and $g(\theta, p, r) := R_C f(\theta, p, r)$ with respect to their angular variables φ and θ can be written as follows:

$$f(t, \varphi, z) = \sum_{l=-\infty}^{\infty} f_l(\rho, z) e^{il\varphi} \quad \text{and} \quad g(\theta, p, r) = \sum_{l=-\infty}^{\infty} g_l(p, r) e^{il\vartheta},$$

where $\theta = (\cos \vartheta, \sin \vartheta)$ and the Fourier coefficients are given by

$$f_l(t, z) = \frac{1}{2\pi} \int_0^{2\pi} f(t, \varphi, z) e^{-il\varphi} d\varphi \quad \text{and} \quad g_l(p, r) = \frac{1}{2\pi} \int_{S^1} g(\theta, p, r) e^{-il\vartheta} d\theta,$$

where $\theta = (\cos \vartheta, \sin \vartheta)$. Consider the l -th Fourier coefficient of the right hand side of (4.6). Then we have

$$\int_{S^1} \int_{\mathbb{R}} \widehat{R_C^* g}(\theta, \xi, \sigma) e^{iR(u-s)\sigma} |\sigma| e^{-il\vartheta} d\sigma d\theta = \int_{\mathbb{R}} \widehat{R_C^* g_l}(\xi, \sigma) e^{iR(u-s)\sigma} |\sigma| d\sigma, \quad (4.8)$$

where $\widehat{R_C^* g_l}$ is the 2-dimensional Fourier transform of $R_C^* g_l$ with respect to (z, ρ) and

$$R_C^* g_l(z, \rho) = \int_{\mathbb{R}} g_l(p, \sqrt{(z-p)^2 + \rho^2}) dp.$$

Theorem 4.1.3. *Let $f \in C_c^\infty(B_R^2 \times \mathbb{R})$. Then we have for $t > 0$*

$$f_l(t, z) = -4i \int_t^\infty \cosh\left(l \operatorname{arccosh} \frac{s}{t}\right) I_2^{-2} R_C^* g_l(z, u-s) \frac{ds}{\sqrt{s^2 - t^2}}.$$

Proof. Applying (1.5) to (4.8) gives

$$\widehat{f}_l(\rho, \xi_1) = -\frac{i}{\pi^2} \int_t^\infty \cosh\left(l \operatorname{arccosh} \frac{s}{t}\right) \int_{\mathbb{R}} \widehat{R_C^* g_l}(\xi) e^{i(u-s)\sigma} |\sigma|^2 d\sigma \frac{ds}{\sqrt{s^2 - t^2}},$$

where \widehat{f}_l is the Fourier transform of f_l with respect to z . □

The regular Radon transform can be obtained from the cylindrical Radon transform.

Theorem 4.1.4. *Let $f \in C_c^\infty(B_R^2 \times \mathbb{R})$. Then we have*

$$\int_{\mathbb{R}} f(t\theta^\perp + (R-s)\theta, z) dt = \frac{2}{\pi} \int_{\mathbb{R}^2} \int_0^\infty sr R_C f(\theta, -\eta, r) e^{-ir^2\xi} e^{-i(2z\eta + (z^2+s^2)+\eta^2)\xi} \xi dr d\eta d\xi.$$

We notice that the expression in the left hand side is the standard 2-dimensional

Radon transform for a fixed z variable. Hence, applying any Radon transform inversion, one gets an inversion of the cylindrical Radon transform R_C .

Proof. Let G be defined by

$$G(\theta, p, \xi) := \int_0^\infty r R_C f(\theta, p, r) e^{-ir^2 \xi} dr.$$

Then we have

$$\begin{aligned} G(\theta, p, \xi) &= \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{R}} \int_{-\pi}^\pi r f(t\theta^\perp + (R - r \cos \psi)\theta, p + r \sin \psi) e^{-ir^2 \xi} d\psi dt dr \\ &= \frac{1}{2\pi} \int f(t\theta^\perp + (R - y)\theta, p + z) e^{-i(y^2 + z^2)\xi} dy dz dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^3} f(t\theta^\perp + (R - y)\theta, z) e^{-i(y^2 + (z-p)^2)\xi} dy dz dt \\ &= \frac{e^{-ip^2 \xi}}{2\pi} \int_{\mathbb{R}^3} f(t\theta^\perp + (R - y)\theta, z) e^{-i(y^2 + z^2)\xi} e^{2ipz\xi} dy dz dt, \end{aligned}$$

where in the second line, we switched from the polar coordinates (r, ψ) to Cartesian coordinates (y, z) . Making the change of variable $r = y^2 + z^2$ yields

$$G(\theta, p, \xi) = \frac{e^{-ip^2 \xi}}{2\pi} \int_0^\infty \int_{\mathbb{R}^2} f(t\theta^\perp + (R - \sqrt{r - z^2})\theta, z) \frac{e^{-ir\xi} e^{2ipz\xi}}{2\sqrt{r - z^2}} dr dz dt,$$

where we do not need to care about $f(t\theta^\perp + (R + \sqrt{r - z^2})\theta, z)$ because f is compactly support on $B_R^2 \times \mathbb{R}$. Let us define the function

$$k_\theta(t, z, r) := \begin{cases} f(t\theta^\perp + (R - \sqrt{r - z^2})\theta, z) / \sqrt{r - z^2} & \text{if } 0 < z^2 < r, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$G(\theta, p, \xi) = \frac{e^{-ip^2\xi}}{4\pi} \int_{\mathbb{R}^3} k_\theta(t, z, r) e^{-ir\xi} e^{2ipz\xi} dr dz dt = \frac{e^{-ip^2\xi}}{4\pi} \int_{\mathbb{R}} \widehat{k}_\theta(t, -2p\xi, \xi) dt,$$

where \widehat{k}_θ is the 2-dimensional Fourier transform of k_θ with respect to the last two variables (z, r) . Also, we have

$$\begin{aligned} \int_{\mathbb{R}} f(t\theta^\perp + (R-s)\theta, z) dt &= \int_{\mathbb{R}} s k_\theta(t, z, z^2 + s^2) dt \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} s \widehat{k}_\theta(t, \eta, \xi) e^{-i(z\eta + (z^2 + s^2)\xi)} dt d\eta d\xi \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} s e^{i\frac{\eta^2}{4\xi}} G\left(\theta, -\frac{\eta}{2\xi}, \xi\right) e^{-i(z\eta + (z^2 + s^2)\xi)} d\eta d\xi \\ &= \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} s G(\theta, -\eta, \xi) e^{-i(2z\eta + (z^2 + s^2) + \eta^2)\xi} \xi d\eta d\xi, \end{aligned}$$

where in the last line, we changed variables $\eta \rightarrow 2\xi\eta$. □

4.1.1.2 Support theorem

By a support theorem, we mean a statement that claims that if integrals of f over all surfaces not intersecting a set A are equal to zero, then f is equal to zero outside A . This statement cannot hold for arbitrary A , but under some geometry restrictions of convexity type.

Lemma 4.1.1. *Let $p_0 \in \mathbb{R}$, $\epsilon > 0$, $B > 0$, and $\theta \in S^1$ be given. Let $f \in C^\infty(B_R^2 \times \mathbb{R})$ and suppose that $g = R_C f$ is equal to zero on the open set $U_{B,\epsilon} = \{(p, r) : |p - p_0| < \epsilon, 0 \leq r < B\}$. Then $\mathcal{R}_\theta f(p, s)$ is equal to zero on the open set $V_B = \{(p, s) :$*

$|p - p_0|^2 + (R - s)^2 < B^2, s > R\}$ where

$$\mathcal{R}_\theta f(p, s) = \int_{\mathbb{R}} f(t\theta^\perp + s\theta, p) dt.$$

We will follow the idea suggested in [7] to prove this lemma.

Proof. Without loss of generality, we may assume $p_0 = 0$. Let $G(\theta, p, r)$ be defined by

$$G(\theta, p, r) = \int_0^r g(\theta, p, s) s ds = \frac{1}{2\pi} \int_{|\eta| \leq r} \int_{\mathbb{R}} f(t\theta^\perp + (R - \eta_1)\theta, p + \eta_2) dt d\eta,$$

where $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Differentiating G with respect to p yields

$$\begin{aligned} \frac{\partial}{\partial p} G(\theta, p, r) &= \frac{1}{2\pi} \int_{|\eta| \leq r} \int_{\mathbb{R}} \frac{\partial}{\partial p} f(t\theta^\perp + (R - \eta_1)\theta, p + \eta_2) dt d\eta \\ &= \frac{1}{2\pi} \int_{|\eta| \leq r} \int_{\mathbb{R}} \frac{\partial}{\partial \eta_2} f(t\theta^\perp + (R - \eta_1)\theta, p + \eta_2) dt d\eta \\ &= \frac{1}{2\pi r} \int_{|\eta|=r} \int_{\mathbb{R}} f(t\theta^\perp + (R - \eta_1)\theta, p + \eta_2) \eta_2 dt d\eta, \end{aligned}$$

where in the last line, we used the divergence theorem. Now we have

$$\begin{aligned} R_C(pf)(\theta, p, r) &= \frac{1}{2\pi r} \int_{\mathbb{R}} \int_{|\eta|=r} (p + \eta_2) f(t\theta^\perp + (R - \eta_1)\theta, p + \eta_2) d\eta dt \\ &= pg(\theta, p, r) + \frac{1}{2\pi} \frac{\partial}{\partial p} G(\theta, p, r) = pg(\theta, p, r) + \frac{\partial}{\partial p} \int_0^r g(\theta, p, s) s ds. \end{aligned}$$

Let the linear operator D be defined by $Dg(\theta, p, r) := pg(\theta, p, r) + \frac{\partial}{\partial p} \int_0^r g(\theta, p, s) s ds$.

Then $R_C(pf)$ is equal to Dg . By iteration, we have $R_C(\mathcal{P}(p)f) = \mathcal{P}(D)g$ where \mathcal{P} is any polynomial. If $g = 0$ in $U_{B,\epsilon}$, then $\mathcal{P}(D)g = 0$ in $U_{B,\epsilon}$. Also, we have for any

point $(p, r) \in U_{B, \epsilon}$,

$$\begin{aligned}
R_C(\mathcal{P}(p)f)(\theta, p, r) &= \frac{1}{2\pi r} \int_{|\eta|=r} \int_{\mathbb{R}} \mathcal{P}(p + \eta_2) f(t\theta^\perp + (R - \eta_1)\theta, p + \eta_2) dt d\eta \\
&= \frac{1}{2\pi} \int_{-r}^r \int_{\mathbb{R}} \mathcal{P}(p + \eta_2) f(t\theta^\perp + (R - \sqrt{r^2 - \eta_2^2})\theta, p + \eta_2) \frac{dt d\eta_2}{\sqrt{r^2 - \eta_2^2}} \\
&= 0.
\end{aligned}$$

For fixed $\theta \in S^1$, $r > 0$, and $p \in \mathbb{R}$, we can choose a sequence of polynomials \mathcal{P}_i such that $\mathcal{P}_i(p + \eta_2)$ converges to $\int_{\mathbb{R}} f(t\theta^\perp + (R - \sqrt{r^2 - \eta_2^2})\theta, p + \eta_2) dt$ uniformly for $|\eta_2| < r$, using The Stone-Weierstrass theorem. It follows that $\mathcal{R}_\theta f(p, s) = 0$ in V_B . \square

Theorem 4.1.5. *Let $p_0 \in \mathbb{R}$ and $B > 0$. Let $f \in C^\infty(B_R^2 \times \mathbb{R})$ and suppose that $g = R_C f$ is equal to zero on the open set $U_B = \{(\theta, p_0, r) : 0 \leq \theta < 2\pi, 0 \leq r < B\}$. Then f is equal to zero on the set $\{(x, y, z) : |(x, y)| > R - \sqrt{B^2 - (p - p_0)^2}, z = p\}$.*

Proof. Let $\epsilon > 0$ be arbitrary. Then g vanishes on the open set $U_{B-\epsilon, \epsilon}$ and by Lemma 4.1.1, $\mathcal{R}_\theta f$ vanishes on the open set $V_{B-\epsilon}$. Let $p \in \mathbb{R}$ be arbitrary. Then by the support theorem of the regular Radon transform [36, 54], f is equal to zero on the set $\{(x, y, z) \in \mathbb{R}^3 : |(x, y)| > R - \sqrt{(B - \epsilon)^2 - (p - p_0)^2}, z = p\}$. \square

Corollary 4.1.1. *Let $A \subset B_R^2 \times \mathbb{R}$ be a closed set invariant under rotation around z -axis and let $f \in C^\infty(B_R^2 \times \mathbb{R})$. Suppose that for any point $(x, y, z) \in \mathbb{R}^3 \setminus A$, there are $(p_{(x,y,z)}, r_{(x,y,z)}) \in \mathbb{R} \times (0, \infty)$ such that a sphere centered at $(Rx/|(x, y)|, Ry/|(x, y)|, p_{(x,y,z)})$ with radius $r_{(x,y,z)}$ separates the point (x, y, z) and A . If $g = R_C f$ vanishes on $\{(\theta, p, r) : p = p_{(x,y,z)}, 0 \leq r < r_{(x,y,z)}, \text{ for any } (x, y, z) \in \mathbb{R}^3 \setminus A\}$, then f vanishes on $\mathbb{R}^3 \setminus A$.*

4.1.1.3 A stability estimate

Here we discuss the stability estimate of the cylindrical Radon transform $R_C f$. For next subsection, we define spaces we need as n -dimension. For $\gamma \geq 0$, let $\mathcal{H}^\gamma(\mathbb{R}^n)$ be the regular Sobolev space with the norm $\|\cdot\|_\gamma$. Let $L_{n-k}^2(S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty))$ be the set of a function g on $S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty)$ with

$$\|g\|^2 := \int_{S^{k-1}} \int_{\mathbb{R}^{n-k}} \int_0^\infty |g(\theta, p, r)|^2 r^{n-k} dr dp d\theta < \infty.$$

Then $L_{n-k}^2(S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty))$ is a Hilbert space. Also, by the Plancherel formula, we have $\|g\| = (2\pi)^{2k-2n-1} \|\tilde{g}\|$, where

$$\tilde{g}(\theta, \xi, |\zeta|) := \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k+1}} g(\theta, p, |w|) e^{-i(p,w) \cdot (\xi, \zeta)} dp dw.$$

Let $\mathcal{H}^\gamma(S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty))$ be the set of a function $g \in L_{n-k}^2(S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty))$ with $\|g\|_\gamma < \infty$, where

$$\|g\|_\gamma^2 := \int_{S^{k-1}} \int_{\mathbb{R}^{n-k}} \int_0^\infty |\tilde{g}(\theta, \xi, \eta)|^2 (1 + |\xi|^2 + |\eta|^2)^\gamma |\eta|^{n-k} d\eta d\xi d\theta.$$

Theorem 4.1.6. *For $\gamma \geq 0$, we have*

$$\|f\|_\gamma \leq 4\pi^{-1} \|R_C f\|_{\gamma+1},$$

for $f \in \mathcal{H}^\gamma(\mathbb{R}^3)$ supported in $B_R^2 \times \mathbb{R}$ (i.e., $n = 3$ and $k = 2$).

Remark 4.1.4. *As mentioned before, R_C can be thought of as the composition of the circular Radon transform and the regular Radon transform. We know that the*

regular Radon transform maps $H^\gamma(\mathbb{R}^2)$ into $H^{\gamma+1/2}(S^1 \times \mathbb{R})$ and a circular Radon transform maps $H^\gamma(\mathbb{R}^2)$ into $H^{\gamma+1/2}(\mathbb{R} \times [0, \infty))$ which is defined by the norm

$$\int_{\mathbb{R}} \int_0^\infty |\tilde{\phi}(\xi, \rho)|^2 (1 + |\xi|^2 + \rho^2)^{\gamma+1/2} \rho d\rho d\xi < \infty$$

in [7, 54]. Hence, in this view, the estimate in Theorem 4.1.6 looks reasonable.

Proof of theorem. Let $g = R_C f$. Note that from (4.5), we have

$$\widehat{R_C^* g}(\theta, \xi, \sigma) = \int_{\mathbb{R}} e^{-i\xi p} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(z, \rho) \cdot (\xi, \sigma)} g(\theta, p, |(z, \rho)|) dz d\rho dp = \tilde{g}(\theta, \xi, |(\xi, \sigma)|).$$

Combining this equation and (4.1), we have

$$\hat{f}(\sigma\theta, \xi) = 4\pi^{-1} \tilde{g}(\theta, \xi, |(\xi, \sigma)|) e^{iR\sigma} |\sigma|.$$

Hence, we have

$$\begin{aligned} \|f\|_\gamma^2 &= \int_{\mathbb{R}^3} (1 + |\iota|^2 + |\xi|^2)^\gamma |\hat{f}(\iota, \xi)|^2 d\iota d\xi \\ &= \int_{\mathbb{R}} \int_{S^1} \int_0^\infty |\sigma| (1 + |\sigma|^2 + |\xi|^2)^\gamma |\hat{f}(\sigma\theta, \xi)|^2 d\sigma d\theta d\xi \\ &= \frac{16}{\pi^2} \int_{S^1} \int_{\mathbb{R}} \int_0^\infty |\sigma|^3 (1 + |(\xi, \sigma)|^2)^\gamma |\tilde{g}(\theta, \xi, |(\xi, \sigma)|)|^2 d\sigma d\xi d\theta \\ &\leq \frac{16}{\pi^2} \int_{S^1} \int_{\mathbb{R}} \int_0^\infty (1 + |(\xi, \sigma)|^2)^\gamma |(\xi, \sigma)|^2 |\tilde{g}(\theta, \xi, |(\xi, \sigma)|)|^2 |\sigma| d\sigma d\xi d\theta \\ &= \frac{16}{\pi^2} \int_{S^1} \int_{\mathbb{R}} \int_{|\xi|}^\infty (1 + |\eta|^2)^\gamma |\eta|^2 |\tilde{g}(\theta, \xi, \eta)|^2 \eta d\eta d\xi d\theta, \end{aligned}$$

where in the last line, we changed the variable $|(\xi, \sigma)|$ to η . □

4.1.1.4 Range conditions

Here we describe the necessary range conditions of the cylindrical Radon transform $R_C f$.

Theorem 4.1.7. *If $g = R_C f$ for $f \in C^\infty(B_R^2 \times \mathbb{R})$, then we have that*

1.

$$\int_{\mathbb{R}} g(\theta, p, \sqrt{(p-z)^2 + (\rho-R)^2}) dp = \int_{\mathbb{R}} g(-\theta, p, \sqrt{(p-z)^2 + (\rho+R)^2}) dp$$

and

2. for $m = 0, 1, 2, \dots$, $\mathcal{P}_z(\theta)$ is a homogeneous polynomial of degree m in θ , where

$$\mathcal{P}_z(\theta) = \int_{\mathbb{R}} I_2^{-1} R_C^* g(\theta, z, R-s) s^m ds.$$

Proof.

1. From (4.1), $I_2^{-1} R_C^* g(\theta, z, R-\rho)$ should be equal to $I_2^{-1} R_C^* g(-\theta, z, R+\rho)$.
2. This follows from (4.6) and the range condition of the regular Radon transform.

□

4.1.2 An n -dimensional case of R_C

In this subsection, we consider the cylindrical Radon transform R_C of a function $f \in C_c^\infty(B_R^k \times \mathbb{R}^{n-k})$ where $n \geq 3$ is arbitrary. As mentioned before (see also [33]), 3-dimensional $R_C f$ can be decomposed into the circular Radon transform and the usual 2-dimensional Radon transform. A natural n -dimensional analog of the cylindrical Radon transform would split into composition of the $n-1$ -dimensional spherical Radon transform and the 2-dimensional Radon transform. We consider

a more general possibility. Namely, $R_{C_{n,k}}f$ of a function $f \in C_c^\infty(B_R^k \times \mathbb{R}^{n-k})$ decomposes into the $n - k + 1$ -dimensional spherical Radon transform and the usual k -dimensional Radon transform. We define $R_{C_{n,k}}f$ for $1 < k \leq n - 1$ and $(\theta, p, r) \in S^{k-1} \times \mathbb{R}^{n-k} \times [0, \infty)$ as follows:

$$R_{C_{n,k}}f(\theta, p, r) = \frac{1}{|S^{n-k}|} \int_{\theta^\perp} \int_{S^{n-k}} f(\tau + (R - r\alpha_1)\theta, p + r\alpha') d\alpha d\tau,$$

where $\alpha = (\alpha_1, \alpha') \in S^{n-k}$ and θ^\perp refers to $\theta^\perp \cap \{(x, z) \in \mathbb{R}^{n-k} \times \mathbb{R}^k : z = p\}$. Then we have an analogue of the Fourier slice theorem, similar to theorem 4.1.1.

Theorem 4.1.8. *Let $f \in C_c^\infty(B_R^k \times \mathbb{R}^{n-k})$. If $g = R_{C_{n,k}}f$, then we have for $(\theta, \xi, \sigma) \in S^{k-1} \times \mathbb{R}^{n-k} \times \mathbb{R}$,*

$$\hat{f}(\sigma\theta, \xi) = 2|S^{n-k}|(2\pi)^{-n+k-1} \widehat{R_{C_{n,k}}^*} g(\theta, \xi, \sigma) e^{iR\sigma} |(\xi, \sigma)|^{n-k-1} |\sigma|,$$

where \hat{f} is the n -dimensional Fourier transform of f and $\widehat{R_{C_{n,k}}^*} g$ is the $n - k + 1$ -dimensional Fourier transform of g in (z, ρ) . Here

$$R_{C_{n,k}}^* g(\theta, z, \rho) = \int_{\mathbb{R}^{n-k}} g(\theta, p, \sqrt{|z - p|^2 + \rho^2}) dp,$$

for $g \in C_c^\infty(S^k \times \mathbb{R}^{n-k} \times [0, \infty))$ and $(z, \rho) \in \mathbb{R}^{n-k} \times \mathbb{R}$.

The proof of this theorem is the almost same as that of theorem 4.1.1. Instead of taking the Hankel transform in r , one take the radial Fourier transform (2.7). Also, we need the identity (2.5). The other steps are the same as in the proof of theorem 4.1.1.

For $\gamma < n - k + 1$, we define the linear operators I^γ and I_2^γ by

$$\widehat{I^\gamma h}(\theta, \xi, \sigma) = |(\xi, \sigma)|^{-\gamma} \hat{h}(\theta, \xi, \sigma) \text{ and } \widehat{I_2^\gamma h}(\theta, \xi, \sigma) = |\sigma|^{-\gamma} \hat{h}(\theta, \xi, \sigma),$$

for a function $h(\theta, z, \rho)$ on $S^{k-1} \times \mathbb{R}^{n-k+1}$ with its $n - k + 1$ -dimensional Fourier transform \hat{h} with respect to (z, ρ) . Then we have the inversion similar to Theorem 4.1.2.

Theorem 4.1.9. *Let $f \in C_c^\infty(B_R^k \times \mathbb{R}^{n-k})$. If $g = R_{C_{n,k}} f$, then we have for $(x, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$,*

$$f(x, z) = \frac{|S^{n-k}|}{(2\pi)^n} \int_{S^{k-1}} I_2^{-k} I^{1-n+k} R_{C_{n,k}}^* g(\theta, z, \rho) \Big|_{\rho=x \cdot \theta - R} d\theta. \quad (4.9)$$

To obtain inversion formula similar to Theorem 4.1.3, let $f(t, \varphi, z)$ be the image function in cylindrical coordinates where $t = |x|$ and $\varphi = x/|x| \in S^{k-1}$. Then the Fourier series of $f(\rho, \phi, z)$ and $g(\theta, p, r)$ with respect to their angular variables can be written as follows:

$$f(t, \varphi, z) = \sum_{l=0}^{\infty} \sum_{j=0}^{N(k,l)} f_{lj}(t, z) Y_{lj}(\varphi) \quad \text{and} \quad g(\theta, p, r) = \sum_{l=0}^{\infty} \sum_{j=0}^{N(k,l)} g_{lj}(p, r) Y_{lj}(\theta),$$

where Y_{lj} is a spherical harmonic and

$$N(k, l) = \frac{(2l + k - 2)(k + l - 3)!}{l!(k - 2)!}, \quad N(k, 0) = 1.$$

From Theorem 4.1.8, we have

$$\int_{\theta^\perp} \hat{f}(\tau + s\theta, \xi) d\tau = \frac{2|S^{n-k}|}{(2\pi)^{n-k+1}} \int_{\mathbb{R}} \widehat{R_{C_{n,k}}^* g}(\theta, \xi, \sigma) e^{i(R-s)\sigma} |\sigma| |(\xi, \sigma)|^{n-k-1} d\sigma. \quad (4.10)$$

Consider the lj -th Fourier coefficient of the right hand side of (4.10). Then we have

$$\begin{aligned} & \int_{S^{k-1}} \int_{\mathbb{R}} \widehat{R_{C_{n,k}}^*} g(\theta, \xi, \sigma) e^{i(R-s)\sigma} |\sigma| |(\xi, \sigma)|^{n-k-1} Y_{lj}(\theta) d\sigma d\theta \\ &= \int_0^\infty \widehat{R_{C_{n,k}}^*} g_{lj}(\xi, \sigma) e^{i(R-s)\sigma} |\sigma| |(\xi, \sigma)|^{n-k-1} d\sigma, \end{aligned} \quad (4.11)$$

Theorem 4.1.10. *Let $f \in C_c^\infty(B_R^k \times \mathbb{R}^{n-k})$. Then we have for $\rho > 0$*

$$f_{lj}(t, z) = \frac{c_k}{(2\pi)^{n-k}} t^{2-k} \int_t^\infty (s^2 - t^2)^{(k-3)/2} C_l^{(k-2)/2} \left(\frac{s}{t}\right) I_2^{-k} I^{1+k-n} R_{C_{n,k}}^* g_{lj}(z, R-s) ds,$$

where

$$c_k = \frac{i^{k-1} 2^{\frac{k}{2}} |S^{n-k}| \Gamma((k-2)/2)}{2\pi^{\frac{n+k+1}{2}} \Gamma(k-2)}.$$

Proof. Applying (4.11) to (1.5) implies that the $n-k$ -dimensional Fourier transform $\widehat{f_{lj}}$ of f_{lj} with respect to z is equal to

$$c_k t^{2-k} \int_t^\infty (s^2 - t^2)^{(k-3)/2} C_l^{(k-2)/2} \left(\frac{s}{t}\right) \int_0^\infty \widehat{R_{C_{n,k}}^*} g_{lj}(\xi, \sigma) e^{i(R-s)\sigma} |\sigma|^k |(\xi, \sigma)|^{n-k-1} d\sigma ds.$$

□

Also, we can get the following theorem similar to Theorem 4.1.4.

Theorem 4.1.11. *Let $f \in C_c^\infty(B_R^k \times \mathbb{R}^{n-k})$. Then we have*

$$\begin{aligned} & \int_{\theta^\perp} f(\tau + s\theta, z) d\tau = \\ & \frac{4|S^{n-k}|}{(2\pi)^{n-k+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-k}} \int_0^\infty (R-s)r R_{C_{n,k}} f(\theta, -p, r) e^{-ir^2\xi} e^{-i(2z \cdot p + (z^2 + (R-s)^2) + |p|^2)\xi} \xi dr dp d\xi. \end{aligned}$$

As in subsubsection 4.1.1, we can obtain a stability estimate and necessary range conditions for $R_{C_{n,k}}$.

Theorem 4.1.12. *For any $f \in \mathcal{H}^\gamma(\mathbb{R}^n)$ supported in $B_R^k \times \mathbb{R}^{n-k}$, we have*

$$\|f\|_\gamma \leq 2|S^{n-k}|(2\pi)^{-n+k-1}\|R_{C_{n,k}}f\|_{\gamma+(n-1)/2}.$$

Also, we have a similar support theorem to Theorem 4.1.5.

Theorem 4.1.13. *Let $p_0 \in \mathbb{R}^{n-k}$ and $B > 0$. Let $f \in C^\infty(B_R^k \times \mathbb{R}^{n-k})$ and suppose that $g = R_{C_{n,k}}f$ is equal to zero on the open set $U_B = \{(\theta, p_0, r) : 0 \leq \theta < 2\pi, 0 \leq r < B\}$. Then f is equal to zero on the set $\{(x, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : |x| > R - \sqrt{B^2 - |p - p_0|^2}, z = p\}$.*

Remark 4.1.5. *We can obtain the same result as Theorem 4.1.7 for an n -dimensional case using Theorem 4.1.8 instead of Theorem 4.1.1.*

4.1.3 Planar geometry

Let us first explain the mathematical model arising in PAT with line detectors introduced in [32]. Let $L_P(\theta, p) = \{(0, y, z) \in \mathbb{R}^3 : (y, z) \cdot \theta = p\}$ for $p > 0$ and $\theta \in S^1$ be a line detector. Then we have $L_P(\theta, p) = L_P(-\theta, -p)$ and a detector line $L_P(\theta, p)$ is located on yz -plane.

Definition 4.1.2. *Let a function f be even in x . The cylindrical Radon transform R_P maps $f \in C_c^\infty(\mathbb{R}^3)$ into*

$$R_P f(\theta, p, r) = \frac{1}{2\pi r} \iint_{d(L_P(\theta, p), (x, y, z))=r} f(x, y, z) d\varpi,$$

for $(\theta, p, r) \in S^1 \times \mathbb{R} \times [0, \infty)$ (see Figure 4.2 (a)). Here $d\varpi$ is the area measure on

the cylinder

$$\{(x, y, z) \in \mathbb{R}^3 : d(L_P(\theta, p), (x, y, z)) = r\}$$

and

$$d(L_P(\theta, p), (x, y, z)) := \sqrt{x^2 + (p - (y, z) \cdot \theta)^2}$$

denotes the Euclidean distance between the line $L_P(\theta, p)$ and the point (x, y, z) .

Remark 4.1.6. We have $R_P f(\theta, p, r) = R_P f(-\theta, -p, r)$.

If f is odd in x , then $R_P f$ is equal to zero. This is the reason why we assume that f is even in x . By definition, we have

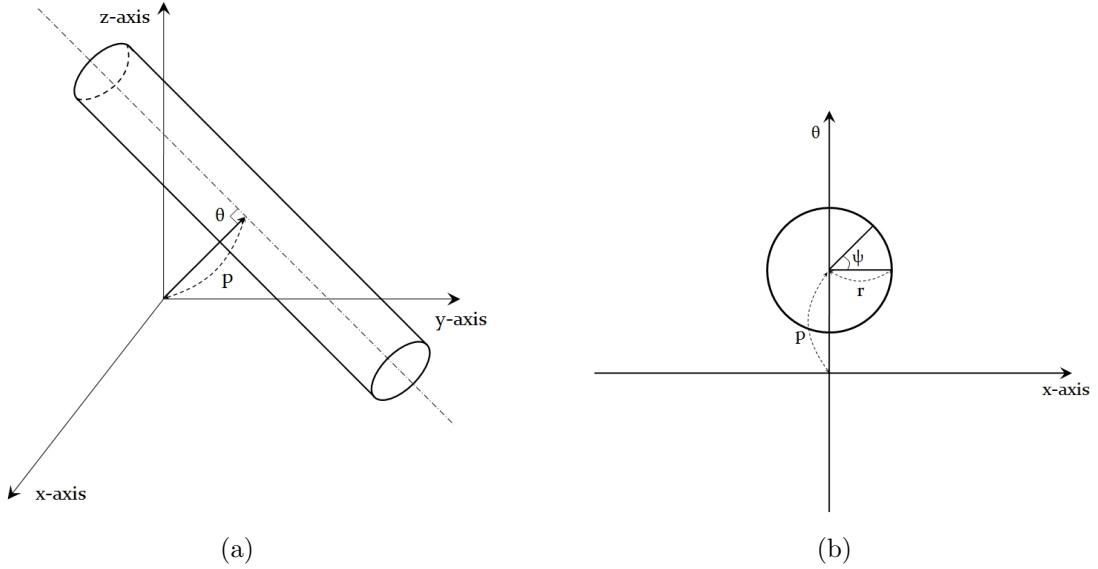


Figure 4.2: (a) the integral area cylinder and (b) the restriction to the $\{(x, t\theta) : x \in \mathbb{R}, t \in \mathbb{R}\}$ plane

$$R_P f(\theta, p, r) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} f(r \cos \psi, t\theta^\perp + (p - r \sin \psi)\theta) d\psi dt,$$

where r is the radius, p and θ are the distance and the direction from the origin to the central axis of the cylinder, t is a parameter along the central axis of the cylinder, and ψ is the polar angle of the circle that is the intersection of plane $\{(x, t\theta) : t \in \mathbb{R}, x \in \mathbb{R}\}$ and the cylinder (see Figure 4.2 (b)).

4.1.3.1 Inversion formulas

We have two integrals in the definition formula of $R_P f$. Like $R_C f$, the inner integral is a circular Radon transform with centers on the line for fixed θ and the outer integral can be thought of as the 2-dimensional regular Radon transform for a fixed x -coordinate [32]. Similarly to the case $R_C f$, we start to apply the inversion of the circular Radon transform for a fixed θ .

We have an analog of the Fourier slice theorem.

Theorem 4.1.14. *Let $f \in C_c^\infty(\mathbb{R}^3)$ be even in x . If $g = R_P f$, then we have*

$$\widehat{f}(\xi, \sigma\theta) = 4|\xi| \widehat{R_P^* g}(\theta, \sigma, \xi), \quad (4.12)$$

where \widehat{f} and $\widehat{R_P^* g}$ are the Fourier transforms of f and $R_P^* g := R_C^* g$ with respect to $(x, y, z) \in \mathbb{R}^3$ and $(z, \rho) \in \mathbb{R}^2$.

Remark 4.1.7. *The evenness of g in (θ, p) implies the evenness of $\widehat{R_P^* g}$ in (θ, σ) .*

Remark 4.1.8. *Equation (4.12) can be thought of as the combination of two Fourier slice theorems: for the circular and regular Radon transforms, too.*

Proof. Taking the Fourier transform of $R_P f$ with respect to p yields

$$\widehat{R_P f}(\theta, \sigma, r) = \frac{1}{\pi} \int_{-1}^1 \widehat{f}(r\sqrt{1-s^2}, \sigma\theta) e^{irs\sigma} \frac{ds}{\sqrt{1-s^2}},$$

where \hat{f} and $\widehat{R_P f}$ are the Fourier transforms of f and $R_P f$ with respect to $(y, z) \in \mathbb{R}^2$ and $p \in \mathbb{R}$, respectively. Taking the Hankel transform H_0 of $\widehat{R_P f}$ with respect to r , we have

$$\begin{aligned}
H_0 \widehat{R_P f}(\theta, \sigma, \eta) &= \frac{1}{\pi} \int_0^\infty \int_{-1}^1 \hat{f}(r\sqrt{1-s^2}, \sigma\theta) e^{irs\sigma} \frac{ds}{\sqrt{1-s^2}} J_0(r\eta) r dr \\
&= \frac{2}{\pi} \int_0^\infty \int_{-1}^1 \hat{f}(r\sqrt{1-s^2}, \sigma\theta) J_0(r\eta) r \cos(rs\sigma) \frac{ds}{\sqrt{1-s^2}} dr \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \hat{f}(b, \sigma\theta) \cos(\rho\sigma) J_0(\eta\sqrt{\rho^2+b^2}) d\rho db,
\end{aligned} \tag{4.13}$$

where in the last line, we made a change of variables $(r, s) \rightarrow (\rho, b)$ where $r = \sqrt{\rho^2 + b^2}$ and $s = \rho/\sqrt{\rho^2 + b^2}$. Applying the identity (4.3) to (4.13), we get

$$H_0 \widehat{R_P f}(\theta, \sigma, \eta) = \begin{cases} \frac{2}{\pi} \int_0^\infty \hat{f}(b, \sigma\theta) \frac{1}{\sqrt{\eta^2 - \sigma^2}} \cos(b\sqrt{\eta^2 - \sigma^2}) db & \text{if } 0 < \sigma < \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting $\eta = \sqrt{\xi^2 + \sigma^2}$ yields

$$H_0 \widehat{R_P f}(\theta, \sigma, |(\sigma, \xi)|) = \frac{2}{\pi} \int_0^\infty \hat{f}(b, \sigma\theta) \frac{\cos(b\xi)}{\xi} db = \frac{1}{\pi} \hat{f}(\xi, \sigma\theta) |\xi|^{-1}. \tag{4.14}$$

As in the proof of theorem 4.1.1, we change the right hand side of (4.14) into a term containing the backprojection operator R_P^* . We have $\widehat{R_P^* g}(\theta, \sigma, \xi) = 2\pi H_0 \hat{g}(\theta, \sigma, |(\sigma, \xi)|)$, so we get (4.12). \square

Let the linear operator I_i for $h \in C_c^\infty(S^1 \times \mathbb{R}^2)$ be defined by $\widehat{I_i^{-1} h}(\theta, \xi) = |\xi_i| \hat{h}(\theta, \xi)$ where \hat{h} is the 2-dimensional Fourier transform of h in the last two dimensional variable. Then we have the following inversion formula.

Theorem 4.1.15. *Let $f \in C_c^\infty(\mathbb{R}^3)$ be even in x . If $g = R_P f$, then we have*

$$f(x, y, z) = 4\pi^{-1} \int_{S^1} I_1^{-1} I_2^{-1} R_P^* g(\theta, \theta \cdot (y, z), x) d\theta.$$

From (4.12), we have

$$\int_{\mathbb{R}} f(x, t\theta^\perp + s\theta) dt = 4I_2^{-1} R_P^* g(\theta, s, x). \quad (4.15)$$

As in below Remark 4.1.9, let $f(x, t, \varphi)$ be the image function in cylindrical coordinates where $(y, z) = t(\cos \varphi, \sin \varphi)$. Consider the l -th Fourier coefficient of the right hand side of (4.15). Then we have

$$\int_{S^1} I_2^{-1} R_P^* g(\theta, s, x) e^{-il\vartheta} d\theta d\vartheta = I_2^{-1} R_P^* g_l(s, x), \quad (4.16)$$

where $\theta = (\cos \vartheta, \sin \vartheta)$. Similarly to Theorem 4.1.3, we have the following theorem:

Theorem 4.1.16. *Let $f \in C_c^\infty(\mathbb{R}^3)$ be even in x . Then we have for $t > 0$*

$$f_l(x, t) = -\frac{2}{\pi} \int_t^\infty (s^2 - t^2)^{-1/2} \cos\left(l \arccos\left(\frac{s}{t}\right)\right) \frac{\partial}{\partial s} I_2^{-1} R_P^* g_l(s, x) ds.$$

Also, we have the following relation between the Radon transform and R_P similar to Theorem 4.1.4.

Theorem 4.1.17. *Let $f \in C_c^\infty(\mathbb{R}^3)$ be even in x . Then we have*

$$\int_{\mathbb{R}} f(x, t\theta^\perp + z\theta) dt = \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty z r R_P f(\theta, -p, r) e^{-ir^2\sigma} e^{-i(2xp + (z^2 + x^2) + p^2)\sigma} \sigma dr dp d\sigma.$$

We can prove this in a way similar to Theorem 4.1.4.

4.1.3.2 A stability estimate

Here we discuss the stability estimate of the cylindrical Radon transform R_P .

For $\gamma \geq 0$, we already define the spaces $\mathcal{H}_e^\gamma(\mathbb{R}^n)$, $L_1^2(S^1 \times \mathbb{R} \times [0, \infty))$, and $H^\gamma(S^1 \times \mathbb{R} \times [0, \infty))$ in subsection 2.4 and subsubsection 4.1.1.3.

Theorem 4.1.18. *For $\gamma \geq 0$, there exists a constant c such that for $f \in \mathcal{H}_e^\gamma(\mathbb{R}^3)$,*

$$\|f\|_\gamma \leq c \|R_P f\|_{\gamma+1}.$$

Proof. Let $g = R_P f$. Similarly to (4.5), we have

$$\widehat{R_P^* g}(\theta, \sigma, \xi) = \int_{\mathbb{R}} e^{-i\sigma p} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(\zeta, \rho) \cdot (\sigma, \xi)} g(\theta, p, |(z, \rho)|) dz d\rho dp = \tilde{g}(\theta, \sigma, |(\sigma, \xi)|). \quad (4.17)$$

Combining this equation and (4.12), we have

$$\hat{f}(\xi, \sigma\theta) = 4|\xi| \tilde{g}(\theta, \sigma, |(\sigma, \xi)|).$$

Hence, we have

$$\begin{aligned} \|f\|_\gamma^2 &= \int_{\mathbb{R}^3} (1 + |\iota|^2 + |\xi|^2)^\gamma |\hat{f}(\xi, \iota)|^2 d\iota d\xi \\ &= 2^{-1} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} |\sigma| (1 + |\sigma|^2 + |\xi|^2)^\gamma |\hat{f}(\xi, \sigma\theta)|^2 d\xi d\sigma d\theta \\ &= 8 \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} |\sigma| (1 + |(\sigma, \xi)|^2)^\gamma |\xi|^2 |\tilde{g}(\theta, \sigma, |(\sigma, \xi)|)|^2 d\xi d\sigma d\theta \\ &= 16 \int_{S^1} \int_{\mathbb{R}} \int_0^\infty |\sigma| (1 + |(\sigma, \xi)|^2)^\gamma |\xi|^2 |\tilde{g}(\theta, \sigma, |(\sigma, \xi)|)|^2 d\xi d\sigma d\theta \\ &= 16 \int_{S^1} \int_{\mathbb{R}} \int_{|\sigma|}^\infty \sqrt{\eta^2 - \sigma^2} |\sigma| (1 + \eta^2)^\gamma |\tilde{g}(\theta, \sigma, \eta)|^2 \eta d\eta d\sigma d\theta, \end{aligned}$$

where in the last line, we changed the variable $|(\sigma, \xi)|$ to η . Continuing the computation yields

$$\|f\|_\gamma^2 \leq c \int_{S^1} \int_{\mathbb{R}} \int_0^\infty (1 + \sigma^2 + \eta^2)^{\gamma+1} |\tilde{g}(\theta, \sigma, \eta)|^2 \eta d\eta d\sigma d\theta.$$

□

4.1.3.3 Range conditions

From Theorem 4.1.14, we have necessary range conditions for R_p as follows:

Theorem 4.1.19. *If $g = R_P f$ for a even function $f \in C^\infty(\mathbb{R}^3)$, then we have*

1. $g(\theta, p, r) = g(-\theta, -p, r)$ and
2. For $m = 0, 1, 2, \dots$, $\mathcal{P}_x(\theta)$ is a homogeneous polynomial of degree m in θ , where

$$\mathcal{P}_x(\theta) = \int_{\mathbb{R}} g(\theta, p, \sqrt{(s-p)^2 + x^2}) s^m ds.$$

Proof.

2. From (4.12) and the range description of the regular Radon transform, we have that for fixed x , the polynomial $\int_{\mathbb{R}} I_2^{-1} R_P^* g(\theta, s, x) s^m ds$ is homogeneous of degree m in θ , which implies that $\mathcal{P}_x(\theta)$ is a homogeneous polynomial of degree m in θ .

□

4.1.4 An n -dimensional case of R_P

As in subsubsection 4.1.2, we consider the cylindrical Radon transform R_P of a function $f \in C_c^\infty(\mathbb{R}^n)$. Assume $n \geq 3$. We define R_{P_n} of a function $f \in C_c^\infty(\mathbb{R}^n)$

even in $x \in \mathbb{R}$ by

$$R_{P_n}f(\theta, p, r) = \frac{1}{2\pi} \int_{\theta^\perp} \int_0^{2\pi} f(r \cos \psi, \tau + (p - r \sin \psi)\theta) d\psi d\tau,$$

for $(\theta, p, r) \in S^{n-2} \times \mathbb{R} \times [0, \infty)$. Here θ^\perp refers to $\theta^\perp \cap \{(0, z) : z \in \mathbb{R}^{n-1}\}$. We still have $R_{P_n}f(\theta, p, r) = R_{P_n}f(-\theta, -p, r)$.

Theorem 4.1.20. *Let $f \in C_c^\infty(\mathbb{R}^n)$ be even in $x \in \mathbb{R}$. If $g = R_{P_n}f$, then we have*

$$\hat{f}(\xi, \sigma\theta) = 4|\xi| \widehat{R_{P_n}^*g}(\theta, \sigma, \xi), \quad (4.18)$$

where \hat{f} and $\widehat{R_{P_n}^*g}$ are the Fourier transforms of f and $R_{P_n}^*g$ with respect to $(x, z) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $(\zeta, \rho) \in \mathbb{R}^2$. Here for a function g on $S^{n-2} \times \mathbb{R} \times [0, \infty)$,

$$R_{P_n}^*g(\theta, \zeta, \rho) = \int_{\mathbb{R}} g(\theta, p, \sqrt{(\zeta - p)^2 + \rho^2}) dp.$$

This proof is similar to that of Theorem 4.1.14. The only difference is that one takes the radial Fourier transform (2.7) and use (2.9) instead of (4.3) as in the proof of Theorem 4.1.8.

Theorem 4.1.21. *Let $f \in C_c^\infty(\mathbb{R}^n)$ be even in x . If $g = R_{P_n}f$, then we have for $(x, z) \in \mathbb{R} \times \mathbb{R}^{n-1}$,*

$$f(x, z) = 2(2\pi)^{2-n} \int_{S^{n-2}} I_1^{2-n} I_2^{-1} R_{P_n}^*g(\theta, \theta \cdot z, x) d\theta.$$

Let $f(x, t, \phi)$ be the image function in cylindrical coordinates where $t = |z|$ and $\varphi = z/|z| \in S^{n-2}$. Then the Fourier series of $f(x, t, \varphi)$ and $g(\theta, p, r)$ with respect to

their angular variables can be written as follows:

$$f(x, t, \varphi) = \sum_{l=0}^{\infty} \sum_{j=0}^{N(n-1,l)} f_{lj}(x, t) Y_{lj}(\varphi) \quad \text{and} \quad g(\theta, p, r) = \sum_{l=0}^{\infty} \sum_{j=0}^{N(n-1,l)} g_{lj}(p, r) Y_{lj}(\theta).$$

From (4.18), we have

$$\int_{\theta^\perp} f(x, \tau + s\theta) d\tau = 4I_2^{-1} R_{P_n}^*(\theta, s, x). \quad (4.19)$$

Consider the lj -th Fourier coefficient of the right hand side of (4.19). Then we have

$$\int_{S^{n-2}} I_2^{-1} R_{P_n}^* g(\theta, s, x) Y_{lj}(\theta) d\theta = I_2^{-1} R_{P_n}^* g_{lj}(s, x). \quad (4.20)$$

Applying (4.20) to (1.5), we have the following theorem.

Theorem 4.1.22. *Let $f \in C_c^\infty(\mathbb{R}^n)$ be even in x . Then we have for $t > 0$*

$$f_{lj}(x, t) = 4c_{n-1} t^{3-n} \int_t^\infty (s^2 - t^2)^{(n-4)/2} C_l^{(n-3)/2} \left(\frac{s}{t} \right) \frac{\partial^{n-2}}{\partial s^{n-2}} I_2^{-1} R_{P_n}^* g_{lj}(s, x) ds,$$

where

$$c_n = \frac{(-1)^{n-1} \Gamma((n-2)/2)}{2\pi^{\frac{n}{2}} \Gamma(n-2)}$$

and

$$R_{P_n}^* g_{lj}(\zeta, \rho) = \int_{\mathbb{R}} g_{lj}(p, \sqrt{(\zeta - p)^2 + \rho^2}) dp.$$

Also, as in subsubsection 4.1.2, we have the following theorem.

Theorem 4.1.23. *Let $f \in C_c^\infty(\mathbb{R}^n)$ be even in x . Then we have*

$$\int_{\theta^\perp} f(x, \tau + s\theta) d\tau = \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty sr R_{P_n} f(\theta, -p, r) e^{-ir^2\sigma} e^{-i(2xp + (s^2 + x^2) + p^2)\sigma} \sigma dr dp d\sigma.$$

As in subsubsection 4.1.3, we can obtain a stability estimate and necessary range conditions for R_{P_n} .

Theorem 4.1.24. *For $\gamma \geq 0$, there exists a constant c_n such that for $f \in \mathcal{H}_e^\gamma(\mathbb{R}^n)$,*

$$\|f\|_\gamma \leq c_n \|R_{P_n} f\|_{\gamma+n-2}.$$

Proof. Let $g = R_{P_n} f$. As in the proof of Theorem 4.1.18, using (4.18), we have

$$\hat{f}(\xi, \sigma\theta) = 4|\xi| \tilde{g}(\theta, \sigma, |(\sigma, \xi)|),$$

so

$$\|f\|_\gamma^2 = 16 \int_{S^{n-2}} \int_{\mathbb{R}} \int_{|\sigma|}^\infty \sqrt{\rho^2 - \sigma^2} |\sigma|^{n-2} (1 + \rho^2)^\gamma |\tilde{g}(\theta, \sigma, \rho)|^2 \rho d\rho d\sigma d\theta.$$

Here, we changed the variable $|(\sigma, \xi)|$ to ρ . Hence, we have

$$\begin{aligned} \|f\|_\gamma^2 &\leq c_n \int_{S^{n-2}} \int_{\mathbb{R}} \int_{|\sigma|}^\infty |\sigma|^{n-2} (\rho^2 - \sigma^2)^{(n-2)/2} (1 + \rho^2)^\gamma |\tilde{g}(\theta, \sigma, \rho)|^2 \rho d\rho d\sigma d\theta \\ &\leq c_n \int_{S^{n-2}} \int_{\mathbb{R}} \int_0^\infty (1 + \sigma^2 + \rho^2)^{\gamma+n-2} |\tilde{g}(\theta, \sigma, \rho)|^2 \rho d\rho d\sigma d\theta. \end{aligned}$$

□

Remark 4.1.9. *Theorem 4.1.19 holds for R_{P_n} for $n \geq 3$.*

4.2 Circular detectors

Some works [83, 84, 85] dealt with PAT with the circular detectors. They showed that the data from PAT with circular detectors is a solution of an initial value problem and used this to reduce this problem to inverting a circular Radon transform. Also, they assume that the circle detectors are centered on a cylinder. In our approach, we define a new Radon-type transform arising in this version of PAT, and consider two situations when the set of the centers of detector circles is a cylinder or a plane. In the next section, we will also study a mathematically similar object, a toroidal Radon transform.

Subsubsection 4.2.1 is devoted to the definition of a Radon-type transform arising in PAT with circular detectors. We reduce this Radon-type transform to the Radon transform on circles with a fixed radius in subsubsection 4.2.2.

4.2.1 A Radon-type transform

In PAT, the acoustic pressure $p(\mathbf{x}, t)$ satisfies the following initial value problem:

$$\begin{aligned} \partial_t^2 p(\mathbf{x}, t) &= \Delta_{\mathbf{x}} p(\mathbf{x}, t) & (\mathbf{x}, t) &\in \mathbb{R}^3 \times (0, \infty), \\ p(\mathbf{x}, 0) &= f(\mathbf{x}) & \mathbf{x} &\in \mathbb{R}^3, \\ \partial_t p(\mathbf{x}, 0) &= 0 & \mathbf{x} &\in \mathbb{R}^3. \end{aligned} \tag{4.21}$$

(We assume that the sound speed in the interior of the object is equal to one.) The goal of PAT is to recover the initial pressure f from measurements of p outside the support of f .

Throughout this subsection, it is assumed that an initial pressure field f is smooth. As mentioned before, we will consider two geometries: the centers of detector circles are located on a cylinder or a plane. In other words, it is assumed that the acoustic signals are measured by a stack of parallel circle detectors where these

circles are centered on a cylinder $B_R^2 \times \mathbb{R}$, or on the $x_1 = 0$ plane, and their radii are a constant r_{det} .

Let $A \subset \mathbb{R}^2$ be the set of x_1 and x_2 coordinates of elements in the set of the centers of the detector circles. Then A becomes a circle in the cylinder case and a line in the plane case (see Figure 4.3). The measured data $P(\mu, z, t)$ for $(\mu, z, t) \in A \times \mathbb{R} \times (0, \infty)$

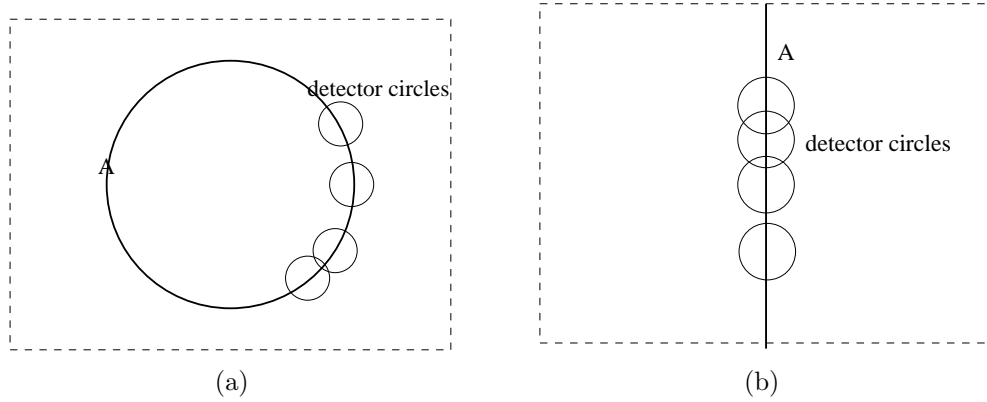


Figure 4.3: Detector circles and a set A (a) A is a circle and (b) A is a line

can be written as

$$P(\mu, z, t) = \frac{1}{2\pi} \int_0^{2\pi} p(\mu + r_{det}\vec{\alpha}, z, t) d\alpha,$$

where $\vec{\alpha} = (\cos \alpha, \sin \alpha)$. Also, it is a well-known fact that

$$p(\mathbf{x}, t) = \partial_t \left(\frac{1}{4\pi t} \int_{\partial B^3(\mathbf{x}, t)} f dS \right),$$

is a solution of PDE system (4.21). Here $B_t^k(\mathbf{x}) = B^k(\mathbf{x}, t)$ is a ball in \mathbb{R}^k centered

at $\mathbf{x} \in \mathbb{R}^k$ with radius t . Hence $P(\mu, z, t)$ becomes

$$\begin{aligned} P(\mu, z, t) &= \frac{1}{2\pi} \int_0^{2\pi} \partial_t \left(\frac{1}{4\pi t} \int_{\partial B^3(\mu + r_{det}\vec{\alpha}, z, t)} f dS \right) d\alpha \\ &= \frac{1}{8\pi^2} \partial_t \left(\frac{1}{t} \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} f(\mu + r_{det}\vec{\alpha} + t \sin \beta_2 \vec{\beta}_1, z + t \cos \beta_2) \sin \beta_2 d\beta_1 d\beta_2 d\alpha \right). \end{aligned}$$

Let us define

$$\mathcal{R}_P f(\mu, z, t) = \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi f(\mu + r_{det}\vec{\alpha} + t \sin \beta_2 \vec{\beta}_1, z + t \cos \beta_2) \sin \beta_2 d\beta_2 d\beta_1 d\alpha.$$

We will demonstrate a relation between the Radon transform on circles with fixed radius-a well studied problem-and $\mathcal{R}_P f$. This allows us to recover f from P .

4.2.2 Reconstruction

Consider the definition of $\mathcal{R}_P f$. The inner integral with respect to β_1 in the definition formula of $\mathcal{R}_P f$ can be thought of as the circular Radon transform with weight $\sin \beta_2$ and centers at the $(\vec{\beta}, z)$ with radius t . We will first remove this integral by applying a similar method which is used in obtaining an inversion formula for the circular Radon transform.

Let us define the operator $\mathcal{R}_P^\#$ for $g \in C_c^\infty(A \times \mathbb{R} \times [0, \infty))$ by

$$\mathcal{R}_P^\# g(\mu, x_3, \rho) = \int_{\mathbb{R}} g(\mu, z, \sqrt{(z - x_3)^2 + \rho^2}) |(z - x_3, \rho)| dz.$$

Let us define the linear operator I_2^{-1} by $\widehat{I_2^{-1}h}(\mu, \xi) = |\xi_2| \hat{h}(\mu, \xi)$, where \hat{h} is the Fourier transform of a function h on $A \times \mathbb{R}^2$ with respect to the last two-dimensional variable.

Lemma 4.2.1. *Let $f \in C_c^\infty(B_R^2(0) \times \mathbb{R})$. Then we have*

$$\int_0^{2\pi} \int_0^{2\pi} f(\mu + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) d\beta_1 d\alpha = -\frac{1}{\pi^2 t} I_2^{-1} \mathcal{R}_P^\# \mathcal{R}_P f(\mu, x_3, t).$$

Proof. By definition, $\mathcal{R}_P f(\mu, z, t)$ can be written as

$$-\int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 f(\mu + r_{det}\vec{\alpha} + t\sqrt{1-s^2}\vec{\beta}_1, z + ts) ds d\beta_1 d\alpha$$

Taking the Fourier transform of $\mathcal{R}_P f$ with respect to z yields

$$\widehat{\mathcal{R}_P f}(\mu, \xi_1, t) = -\int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 \hat{f}(\mu + r_{det}\vec{\alpha} + t\sqrt{1-s^2}\vec{\beta}_1, \xi_1) e^{its\xi_1} ds d\beta_1 d\alpha,$$

where \hat{f} and $\widehat{\mathcal{R}_P f}$ are the 1-dimensional Fourier transforms of f and $\mathcal{R}_P f$ with respect to x_3 and z , respectively. Taking the Hankel transform of order zero of $t\widehat{\mathcal{R}_P f}$ with respect to t , we conclude that $H_0(t\widehat{\mathcal{R}_P f})(\mu, \xi_1, \eta)$ is equal to

$$\begin{aligned} & -\int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 \hat{f}(\mu + r_{det}\vec{\alpha} + t\sqrt{1-s^2}\vec{\beta}_1, \xi_1) e^{its\xi_1} ds d\beta_1 d\alpha t^2 J_0(t\eta) dt \\ &= -2 \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_0^1 \hat{f}(\mu + r_{det}\vec{\alpha} + t\sqrt{1-s^2}\vec{\beta}_1, \xi_1) t^2 J_0(t\eta) \cos(ts\xi_1) ds d\beta_1 d\alpha dt \\ &= -2 \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^\infty \hat{f}(\mu + r_{det}\vec{\alpha} + b\vec{\beta}_1, \xi_1) b \cos(\rho\xi_1) J_0(\eta\sqrt{\rho^2 + b^2}) d\rho db d\beta_1 d\alpha, \end{aligned}$$

where in the last line, we changed variables $(t, s) \rightarrow (\rho, b)$ where $t = \sqrt{\rho^2 + b^2}$ and

$s = \rho/\sqrt{\rho^2 + b^2}$. Using the identity (4.3), we get

$$H_0(\widehat{t\mathcal{R}_P f})(\mu, \xi_1, \eta) = -2 \begin{cases} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \hat{f}(\mu + r_{det}\vec{\alpha} + b\vec{\beta}_1, \xi_1) b \frac{\cos(b\sqrt{\eta^2 - \xi_1^2})}{\sqrt{\eta^2 - \xi_1^2}} db d\beta_1 d\alpha & \text{if } 0 < \xi_1 < \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting $\eta = \sqrt{\xi_1^2 + \xi_2^2}$ yields

$$H_0(\widehat{t\mathcal{R}_P f})(\mu, \xi_1, |\xi|) = -2 \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \hat{f}(\mu + r_{det}\vec{\alpha} + b\vec{\beta}_1, \xi_1) \frac{b}{\xi_2} \cos(b\xi_2) db d\beta_1 d\alpha. \quad (4.22)$$

The inner integral in the right hand side of the last equation is the Fourier cosine transform with respect to b , so taking the Fourier cosine transform of (4.22), we get

$$\int_0^{2\pi} \int_0^{2\pi} \hat{f}(\mu + r_{det}\vec{\alpha} + s\vec{\beta}_1, \xi_1) s d\beta_1 d\alpha = -\pi^{-1} \int_0^\infty H_0(\widehat{t\mathcal{R}_P f})(\mu, \xi_1, |\xi|) \cos(s\xi_2) \xi_2 d\xi_2, \quad (4.23)$$

where \hat{f} is the Fourier transform of f with respect to x_3 .

As in the proof of Theorem 4.1.1, we get $\widehat{\mathcal{R}_P^\# g}(\mu, \xi) = 2\pi H_0(t\hat{g})(\mu, \xi_1, |\xi|)$ where $\widehat{\mathcal{R}_P^\# g}$ is the Fourier transform with respect to the last variable (x_3, ρ) . Combining this with (4.23), we have for $g = \mathcal{R}_P f$,

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \hat{f}(\mu + r_{det}\vec{\alpha} + s\vec{\beta}_1, \xi_1) d\beta_1 d\alpha &= -\frac{1}{2\pi^2 s} \int_0^\infty \widehat{\mathcal{R}_P^\# g}(\mu, \xi) \cos(s\xi_2) \xi_2 d\xi_2 \\ &= -\frac{1}{\pi^2 s} \int_{\mathbb{R}} \widehat{\mathcal{R}_P^\# g}(\mu, \xi) e^{is\xi_2} |\xi_2| d\xi_2. \end{aligned}$$

□

Let a Radon-type transform $M_{r_{det}}f$ be defined by

$$M_{r_{det}}f(\mathbf{x}) := \int_0^{2\pi} f(r_{det}\vec{\alpha} + (x_1, x_2), x_3)d\alpha,$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. We will show that $M_{r_{det}}f$ can be obtained from $\mathcal{R}_P f$ when A is a line or circle.

Remark 4.2.1. *When we have two circular detectors with different radii, say r_1, r_2 , we have two different values $M_{r_1}f, M_{r_2}f$ for each \mathbf{x} , i.e., two Radon transforms on circles with two different fixed radii. Some works [11, 69, 81] show how f can be reconstructed from $M_{r_1}f, M_{r_2}f$ under a certain assumption.*

The following theorem describes an inversion for the Radon transform $M_{r_{det}}$ over all circles with a fixed radius.

Theorem 4.2.1. *Let $f \in C_c^\infty(\mathbb{R}^3)$. Then we have*

$$\hat{f}(\rho, \theta, x_3) = \frac{r_{det}^n}{2\pi} \sum_{n=0}^{\infty} \rho^{-n} \int_0^{\rho} \rho_{n-1} \int_0^{\rho_{n-1}} \rho_{n-2} \cdots \int_0^{\rho_1} \rho_0 \widehat{M_{r_{det}}f}(\rho_0, \theta, x_3) d\rho_0 \cdots d\rho_{n-2} d\rho_{n-1}.$$

Proof. Let us take the Fourier transform of $M_{r_{det}}f$ with respect to (x_1, x_2) . Then we have $\widehat{M_{r_{det}}f}(\xi, x_3) = 2\pi \hat{f}(\xi, x_3) J_0(r_{det}|\xi|)$ or $\widehat{M_{r_{det}}f}(\rho, \theta, x_3) = 2\pi \hat{f}(\rho, \theta, x_3) J_0(r_{det}\rho)$ where $\rho = |\xi|$, $\theta = \xi/|\xi|$, and \hat{f} and $\widehat{M_{r_{det}}f}$ are the 2-dimensional Fourier transforms of f and $M_{r_{det}}f$ with respect to (x_1, x_2) . We used the following identities:

$$\int_0^{\rho} r^n J_{n-1}(r) dr = \rho^n J_n(\rho), \quad \text{and} \quad \sum_{n=0}^{\infty} J_n(\rho) = 1 \text{ for any } \rho \quad (4.24)$$

(see e.g. [29]). Integrating $\widehat{M_{r_{det}}f}(\rho, \theta, x_3)r_{det}\rho$ with respect to ρ yields

$$\rho^{-1} \int_0^\rho r_{det}\rho_0 \widehat{M_{r_{det}}f}(\rho_0, \theta, x_3)\rho_0 d\rho_0 = 2\pi \hat{f}(\rho, \theta, x_3)J_1(r_{det}\rho).$$

Inductively, we get

$$\begin{aligned} & 2\pi \hat{f}(\rho, \theta, x_3)J_n(r_{det}\rho) \\ &= \rho^{-n} \int_0^\rho r_{det}\rho_{n-1} \int_0^{\rho_{n-1}} r_{det}\rho_{n-2} \cdots \int_0^{\rho_1} r_{det}\rho_0 \widehat{M_{r_{det}}f}(\rho_0, \theta, x_3) d\rho_0 \cdots d\rho_{n-2} d\rho_{n-1}. \end{aligned}$$

Equation (4.24) completes this proof. \square

4.2.2.1 Cylindrical geometry

Let $B_R^2(0) \times \mathbb{R}$ be the solid cylinder $\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq R\}$. Let the centers of the detector circles be located on the cylinder $\partial B_R^2(0) \times \mathbb{R} = A \times \mathbb{R}$. Then A is a circle centered at the origin with radius R . Also, μ can be represented as $R\vec{\theta}$ for $\theta \in [0, 2\pi)$. Then Lemma 4.2.1 implies that

$$\int_0^{2\pi} \int_0^{2\pi} f(R\vec{\theta} + r_{det}\vec{\alpha} + t\beta_1, x_3) d\beta_1 d\alpha = -\frac{1}{\pi^2 t} I_2^{-1} \mathcal{R}_p^\# R_p f(\mu, x_3, t). \quad (4.25)$$

Again, the inner integral with respect to β_1 in the left hand side of (4.25) is the circular Radon transform with centers on $\partial B_R^2(0)$ and radius t . Hence, if applying an inversion formula of the circular Radon transform, we get $M_{r_{det}}f(\mathbf{x})$.

Theorem 4.2.2. *Let f be a smooth function supported in $B_R^2(0) \times \mathbb{R}$. Then for any*

$\mathbf{x} \in \mathbb{R}^3$, $M_{r_{det}} f(\mathbf{x})$ is equal to

$$-\frac{1}{2\pi^2 R} \Delta_{x_1, x_2} \int_0^{2\pi} \int_0^{2(R+r_{det})} I_2^{-1} \mathcal{R}_P^\# \mathcal{R}_P f(R\theta, x_3, t) \log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right| dt d\theta.$$

To prove this theorem, we follow the method discussed in [23].

Proof. It is computed in [23] that

$$\int_0^{2\pi} \log \left| |(x_1, x_2) - R\vec{\theta}|^2 - |(y_1, y_2) - R\vec{\theta}|^2 \right| d\theta = 2\pi R \log |(x_1, x_2) - (y_1, y_2)| + 2\pi R \log R.$$

For any measurable function g on \mathbb{R} , it is easily shown that

$$\int_0^{2(R+r_{det})} t \int_0^{2\pi} \int_0^{2\pi} f(R\vec{\theta} + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) d\beta_1 d\alpha q(t) dt = \int_0^{2\pi} \int_{\mathbb{R}^2} f(R\vec{\theta} + r_{det}\vec{\alpha} + w, x_3) q(|w|) dw d\alpha.$$

Applying this with $q(t) = \log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right|$ and making the change of variables $(y_1, y_2) = R\vec{\theta} + t\vec{\beta}_1 \in \mathbb{R}^2$ give

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2(R+r_{det})} \int_0^{2\pi} \int_0^{2\pi} t f(R\vec{\theta} + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) \log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right| d\beta_1 d\alpha dt d\theta \\ &= \int_0^{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} f(r_{det}\vec{\alpha} + (y_1, y_2), x_3) \log \left| |(x_1, x_2) - R\vec{\theta}|^2 - |(y_1, y_2) - R\vec{\theta}|^2 \right| dy d\alpha d\theta \end{aligned}$$

Hence, we have

$$= 2\pi R \int_0^{2\pi} \int_{\mathbb{R}^2} f(r_{det}\vec{\alpha} + (y_1, y_2), x_3) (\log |(x_1, x_2) - (y_1, y_2)| + \log R) dy d\alpha,$$

where in the last line, we used the Fubini-Tonelli theorem. Since $\log |(x_1, x_2) -$

$(y_1, y_2)| + \log R$ is a fundamental solution of the Laplacian in \mathbb{R}^2 , we have

$$\begin{aligned} & \int_0^{2\pi} f((x_1, x_2) + r_{det}\vec{\alpha}, x_3) d\alpha = \\ & \frac{1}{2R} \Delta_{x_1, x_2} \int_0^{2\pi} \int_0^{2(R+r_{det})} \int_0^{2\pi} \int_0^{2\pi} t f(R\vec{\theta} + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) \log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right| d\beta_1 d\alpha dt d\theta. \end{aligned}$$

Lemma 4.2.1 completes this proof. \square

When A is a circle, we can reconstruct f from $\mathcal{R}_P f$ by applying Theorems 4.2.1 and 4.2.2.

4.2.2.2 Planar geometry

Let the centers of detector circles be located on the $x_1 = 0$ plane. Then $A \subset \mathbb{R}^2$ is the $x_1 = 0$ line. Also, $\mathcal{R}_P f$ is equal to zero if f is an odd function in x_1 . We thus assume the function is even in x_1 . We will denote $\mu \in A$ by $(0, \mu) \in \mathbb{R}^2$ (notation abuse). Then Lemma 4.2.1 implies that

$$\int_0^{2\pi} \int_0^{2\pi} f((0, \mu) + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) d\beta_1 d\alpha = -\frac{1}{\pi^2 t} I_2^{-1} \mathcal{R}_p^\# \mathcal{R}_p f(\mu, x_3, t). \quad (4.26)$$

Again, the inner integral with respect to β_1 in the left hand side of (4.26) is the circular Radon transform with centers on the line $(0, \mu)$. Hence, by applying an inversion formula of the circular Radon transform, we get $M_{r_{det}} f(\mathbf{x})$ and radius t .

Theorem 4.2.3. *Let $f \in C_c^\infty(\mathbb{R}^2 \times [0, \infty))$ be an even function in x_1 . Then we have*

$$M_{r_{det}} f(\mathbf{x}) = \frac{1}{2\pi^4} \int_{\mathbb{R}} \int_{\mathbb{R}} I_2^{-1} \mathcal{R}_P^\# \mathcal{R}_P f(\mu, x_3, \sqrt{s^2 + (x_2 - \mu)^2}) \frac{d\mu ds}{\sqrt{s^2 + (x_2 - \mu)^2} |x_1 - s|^2}.$$

Remark 4.2.2. *The pressure is measured outside of the object and the circular*

detectors are located on the xy -plane. We thus assume $f \in C_c^\infty(\mathbb{R}^2 \times [0, \infty))$.

To prove this theorem, we follow the method in [7, 21, 55, 57].

Proof. In this proof, we fix x_3 . Let us define Mf as

$$Mf(\mu, t) = \int_0^{2\pi} \int_0^{2\pi} f((0, \mu) + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) d\beta_1 d\alpha.$$

Then the inner integration with respect to β_1 is the circular Radon transform with centers at the line $(0, \mu)$ for fixed x_3 and α . Also, Lemma 4.2.1 implies $Mf(\mu, t) = -(\pi^2 s)^{-1} I_2^1 \mathcal{R}_P^\# \mathcal{R}_P f(\mu, x_3, s)$. By definition, $Mf(\mu, t)$ can be written as

$$2 \int_0^{2\pi} \int_{-1}^1 f((0, \mu) + r_{det}\vec{\alpha} + t(\sqrt{1-s^2}, s), x_3) \frac{ds}{\sqrt{1-s^2}} d\alpha$$

Taking the Fourier transform of Mf with respect to μ yields

$$\widehat{Mf}(\xi_1, t) = 2 \int_0^{2\pi} \int_{-1}^1 \hat{f}(r_{det} \cos \alpha + t\sqrt{1-s^2}, \xi_1, x_3) e^{i(ts+r_{det} \sin \alpha)\xi_1} \frac{ds}{\sqrt{1-s^2}} d\alpha,$$

where \hat{f} and \widehat{Mf} are the 1-dimensional Fourier transforms of f and Mf with respect to x_2 and μ , respectively. Taking the Hankel transform of order zero of \widehat{Mf} with respect to t , we conclude that $H_0(Mf)(\mu, \xi_1, |\xi|)$ is equal to

$$\begin{aligned} & 2 \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 \hat{f}(r_{det} \cos \alpha + t\sqrt{1-s^2}, \xi_1, x_3) e^{i(ts+r_{det} \sin \alpha)\xi_1} \frac{ds}{\sqrt{1-s^2}} d\alpha t J_0(t|\xi|) dt \\ & 4 \int_0^{2\pi} \int_0^\infty \int_0^\infty \hat{f}(r_{det}\vec{\alpha} + b, \xi_1, x_3) \cos(\rho\xi_1) J_0(|\xi| \sqrt{\rho^2 + b^2}) e^{i\xi_1 r_{det} \sin \alpha} d\rho db d\alpha, \end{aligned}$$

where in the last line, we made a change of variables $(t, s) \rightarrow (\rho, b)$ where $t =$

$\sqrt{\rho^2 + b^2}$ and $s = \rho/\sqrt{\rho^2 + b^2}$.

Using the identity (4.3), we get

$$H_0 \widehat{Mf}(\xi_1, |\xi|) = 4 \int_0^{2\pi} \int_0^\infty \hat{f}(r_{det} \cos \alpha + b, \xi_1, x_3) \frac{\cos(b\xi_2)}{\xi_2} e^{i\xi_1 r_{det} \sin \alpha} db d\alpha. \quad (4.27)$$

Again, the inner integral in the right hand side of the last equation is the Fourier cosine transform with respect to b , so taking the Fourier cosine transform of (4.27), we get

$$\int_0^{2\pi} \hat{f}(r_{det} \cos \alpha + s, \xi_1, x_3) e^{i\xi_1 r_{det} \sin \alpha} d\alpha = 2\pi^{-1} \int_0^\infty H_0 \widehat{Mf}(\xi_1, |\xi|) \cos(s\xi_2) \xi_2 d\xi_2, \quad (4.28)$$

where \hat{f} is the Fourier transform of f with respect to the variable x_2 .

Let us define the operator M^* for $g \in C_c^\infty(\mathbb{R} \times [0, \infty))$ by

$$M^*g(x_2, x_1) = \int_{\mathbb{R}} g(\mu, \sqrt{x_1^2 + (x_2 - \mu)^2}) d\mu.$$

We can change $H_0 \widehat{Mf}(\xi_1, |\xi|)$ into $\widehat{M^*g}(\xi)$ similarly to (4.5), where $\widehat{M^*g}$ is the Fourier transform of M^*g with respect to the variables (x_2, x_1) .

$$\int_0^{2\pi} \hat{f}(r_{det} \cos \alpha + s, \xi_1, x_3) e^{i\xi_1 r_{det} \sin \alpha} d\alpha = \frac{1}{2\pi^2} \int_{\mathbb{R}} M^*g(\xi) e^{is\xi_2} |\xi_2| d\xi_2,$$

Thus, we have

$$\begin{aligned}
\int_0^{2\pi} f(r_{det}\vec{\alpha} + (x_1, x_2), x_3) d\alpha &= \frac{1}{4\pi^3} \int_{\mathbb{R}} \int_{\mathbb{R}} M^*g(x_2, s) e^{i(x_1-s)\xi_2} |\xi_2| ds d\xi_2 \\
&= -\frac{1}{2\pi^2} \int_{\mathbb{R}} M^*g(x_2, s) |x_1 - s|^{-2} ds.
\end{aligned}$$

□

When A is a line, we can determine f from $\mathcal{R}_P f$ applying Theorems 4.2.1 and 4.2.3.

5. THE TOROIDAL RADON TRANSFORM

As mentioned before, circular detectors as well as line detectors have been used to measure the acoustic data in PAT. PAT with line detectors reduces to the cylindrical Radon transform [32, 33]. The detector line corresponds to the central axis of a cylindrical and the propagation distance of the acoustic wave gives the radius of a cylinder. PAT with circular detectors suggests to consider a toroidal Radon transform, which assigns a given locally integrable function to its integrals over a set of tori. If we think that the detector circle corresponds to the central circle of a torus and the propagation distance of the acoustic wave gives the radius of the tube of this torus, then it looks reasonable that PAT with circular detectors brings about the toroidal Radon transform. Unfortunately, we have not been able to establish the direct link between PAT with circular detectors and the toroidal Radon transform. Nevertheless, studying the toroidal Radon transform seems to be an interesting geometric object in its own right.

In this section, we study the toroidal Radon transform with the centers of tori located on a cylinder or on a plane. Our goal is to present inversion formulas for these cases.

Subsubsection 5.1 is devoted to the definition of the toroidal Radon transform. Various inversion formulas are provided in subsection 5.2.

5.1 Definition of the toroidal Radon transform

Definition 5.1.1. *Let $u > 0$ be a radius of the central circles of tori. Let $A \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$ be the set of the centers of tori. The toroidal Radon transform R_T maps*

$f \in C_c^\infty(\mathbb{R}^3)$ into

$$R_T f(\mu, p, r) = \frac{1}{2\pi} \int_{S^1} \int_0^{2\pi} f(\mu + (u - r \cos \beta) \vec{\alpha}, p + r \sin \beta) d\beta d\vec{\alpha}, \quad (5.1)$$

for $(\mu, p, r) \in A \times \mathbb{R} \times (0, \infty)$. Here $\vec{\alpha} = (\cos \alpha, \sin \alpha)$, α is the angular parameter along the central circle, (μ, p) is the center of the torus, and β and r are the polar angle and radius of the tube of the torus, respectively.

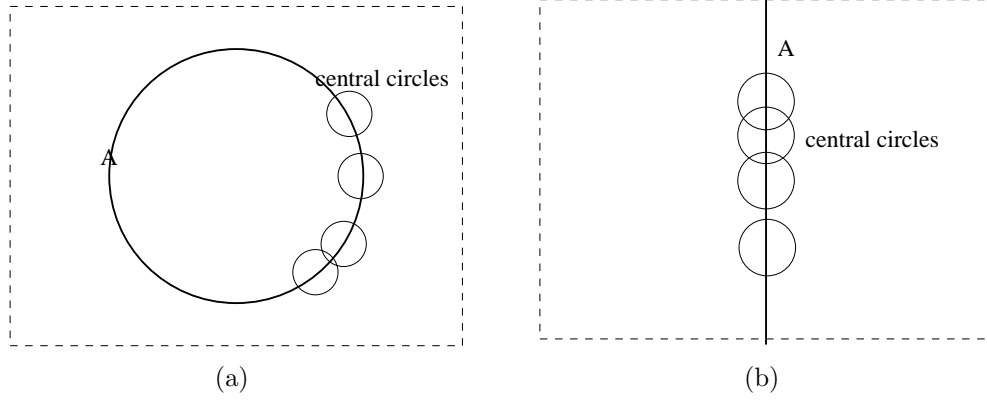


Figure 5.1: Central circles and a set A (a) A is a circle and (b) A is a line

We consider the two situations when A is a circle or a line and thus the set of the centers of tori is a cylinder or a plane (see Figure 5.1). We then present the relation between the circular Radon transform and the toroidal Radon transform. This relation leads naturally to an inversion formula, if one uses an inversion formula for the circular Radon transform (already discussed in [23, 48] or [7, 21, 55, 57, 61]).

Definition 5.1.2. Let f be a compactly supported function in \mathbb{R}^3 . The circular

Radon transform M maps a function f into

$$Mf(\mu, x_3, r) = \int_{S^1} f(\mu + r\vec{\alpha}, x_3) d\vec{\alpha} \quad \text{for } (\mu, x_3, r) \in A \times \mathbb{R} \times (0, \infty).$$

5.2 Inversion of the toroidal Radon transform

The inner integral with respect to β in (5.1) can be thought of as the circular Radon transform with centers at $(\mu + u\vec{\alpha}, p)$ and radius r . As in subsection 4.1, we will first invert this transform.

Let us define the operator R_T^* for $g \in C_c^\infty(A \times \mathbb{R} \times [0, \infty))$ by

$$R_T^*g(\mu, z, \rho) = \int_{\mathbb{R}} g(\mu, p, \sqrt{(z-p)^2 + \rho^2}) dp,$$

where $(\mu, z, \rho) \in A \times \mathbb{R}^2$. The following theorem shows the relation between the circular and toroidal Radon transforms.

Lemma 5.2.1. *Let $f \in C_c^\infty(\mathbb{R}^3)$. Then we have*

$$\frac{1}{2}I_2^{-1}R_T^*R_Tf(\mu, x_3, r) = \begin{cases} Mf(\mu, x_3, u-r) + Mf(\mu, x_3, u+r) & \text{if } u > r, \\ Mf(\mu, x_3, r-u) + Mf(\mu, x_3, u+r) & \text{otherwise.} \end{cases} \quad (5.2)$$

Proof. By definition, we have

$$R_Tf(\mu, p, r) = \frac{1}{2\pi} \sum_{j=1}^2 \int_{S^1} \int_{-1}^1 f(\mu + (u + (-1)^j r \sqrt{1-s^2})\vec{\alpha}, p + rs) \frac{ds}{\sqrt{1-s^2}} d\vec{\alpha}.$$

We take the Fourier transform of R_Tf with respect to p and the Hankel transform

of order zero of $\widehat{R_T f}$ with respect to r . Then we have

$$H_0 \widehat{R_T f}(\mu, \xi_1, \eta) = \frac{1}{2\pi} \sum_{j=1}^2 \int_{S^1} \int_0^\infty \int_0^\infty \hat{f}(\mu + (u + (-1)^j b)\vec{\alpha}, \xi_1) \cos(\rho \xi_1) J_0(\eta \sqrt{\rho^2 + b^2}) d\rho db d\vec{\alpha}, \quad (5.3)$$

where \hat{f} and $\widehat{R_T f}$ are correspondingly the 1-dimensional Fourier transforms of f and $R_T f$ with respect to z and p . Lastly, we change variables $(r, s) \rightarrow (\rho, b)$, where $r = \sqrt{\rho^2 + b^2}$ and $s = \rho/\sqrt{\rho^2 + b^2}$. Applying (4.3) to (5.3), we get

$$H_0 \widehat{R_T f}(\mu, \xi_1, |\xi|) = \frac{1}{2\pi} \sum_{j=1}^2 \int_{S^1} \int_0^\infty \hat{f}(\mu + (u + (-1)^j b)\vec{\alpha}, \xi_1) \frac{\cos(b\xi_2)}{\xi_2} db d\vec{\alpha}.$$

The inner integral in the right side of the last equation is the Fourier cosine transform with respect to b , so taking the inverse Fourier cosine transform of the above formula, we get

$$\sum_{j=1}^2 \int_{S^1} \hat{f}(\mu + (u + (-1)^j s)\vec{\alpha}, \xi_1) d\vec{\alpha} = 4 \int_0^\infty H_0 \widehat{R_T f}(\mu, \xi_1, |\xi|) \cos(s\xi_2) \xi_2 d\xi_2. \quad (5.4)$$

For a fixed ξ_1 , one recognizes the sum of two circular Radon transforms on the left.

Similarly to (4.5), we can change the right hand side of (5.4) into a term containing operator R_T^* , i.e.,

$$\widehat{R_T^* g}(\mu, \xi) = 2\pi H_0 \hat{g}(\mu, \xi_1, |\xi|). \quad (5.5)$$

Here $\widehat{R_T^* g}$ is the Fourier transform with respect to the variables (z, ρ) . Combining

this with (5.4), we have for $g = R_T f$,

$$\begin{aligned} \sum_{j=1}^2 \int_{S^1} \hat{f}(\mu + (u + (-1)^j s)\vec{\alpha}, \xi_1) d\vec{\alpha} &= \frac{2}{\pi} \int_0^\infty \widehat{R_T^* g}(\mu, \xi) \cos(s\xi_2) \xi_2 d\xi_2 \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \widehat{R_T^* g}(\mu, \xi) e^{is\xi_2} |\xi_2| d\xi_2, \end{aligned}$$

since $\widehat{R_T^* g}$ is even in ξ_2 . □

The following lemma describes another relation between the toroidal and circular Radon transforms.

Lemma 5.2.2. *Let $f \in C_c^\infty(\mathbb{R}^3)$. Then we have*

$$\begin{aligned} &\frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty r s R_T f(\mu, -\eta, s) e^{-i(s^2 + 2x_3\eta + x_3^2 - \eta^2 + r^2)\xi} \xi ds d\eta d\xi \\ &= \begin{cases} Mf(\mu, x_3, u - r) + Mf(\mu, x_3, u + r) & \text{if } u > r, \\ Mf(\mu, x_3, r - u) + Mf(\mu, x_3, u + r) & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Let G be defined by

$$G(\mu, p, \xi) := \int_0^\infty r R_T f(\mu, p, r) e^{-ir^2\xi} dr.$$

Then we have

$$\begin{aligned} G(\mu, p, \xi) &= \frac{1}{2\pi} \int_0^\infty \int_{S^1} \int_{-\pi}^\pi r f(\mu + (u - r \cos \beta)\vec{\alpha}, p + r \sin \beta) e^{-ir^2\xi} d\beta d\vec{\alpha} dr \\ &= \frac{1}{2\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\mu + (u - y)\vec{\alpha}, p + z) e^{-i(y^2 + z^2)\xi} dy dz d\vec{\alpha}, \end{aligned}$$

where we switched from the polar coordinates (r, β) to the Cartesian coordinates

$(y, z) \in \mathbb{R}^2$. Continuing the computation yields

$$\begin{aligned} G(\mu, p, \xi) &= \frac{1}{2\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\mu + (u - y)\vec{\alpha}, z) e^{-i(y^2 + (z-p)^2)\xi} dy dz d\vec{\alpha} \\ &= \frac{e^{-ip^2\xi}}{2\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\mu + (u - y)\vec{\alpha}, z) e^{-i(y^2 + z^2)\xi} e^{2ipz\xi} dy dz d\vec{\alpha}, \end{aligned}$$

Making the change of variables $r = y^2 + z^2$ yields

$$G(\mu, p, \xi) = \frac{e^{-ip^2\xi}}{2\pi} \sum_{j=1}^2 \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\mu + (u + (-1)^j \sqrt{r - z^2})\vec{\alpha}, z) \frac{e^{-ir\xi} e^{2ipz\xi}}{2\sqrt{r - z^2}} dr dz d\vec{\alpha}.$$

Let us define the function

$$k_\mu(\alpha, z, r) := \begin{cases} \sum_{j=1}^2 f(\mu + (u + (-1)^j \sqrt{r - z^2})\vec{\alpha}, z) / \sqrt{r - z^2} & \text{if } 0 < z^2 < r, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$G(\mu, p, \xi) = \frac{e^{-ip^2\xi}}{4\pi} \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}} k_\mu(\alpha, z, r) e^{-ir\xi} e^{2ipz\xi} dr dz d\vec{\alpha} = \frac{e^{-ip^2\xi}}{4\pi} \int_{S^1} \widehat{k}_\mu(\alpha, -2p\xi, \xi) d\vec{\alpha}, \quad (5.6)$$

where \widehat{k}_μ is the 2-dimensional Fourier transform of k_μ with respect to the variables (z, r) . Also, we have

$$\begin{aligned} \sum_{j=1}^2 \int_{S^1} f(\mu + (u + (-1)^j s)\alpha, x_3) d\alpha &= \int_{S^1} s k_\mu(\alpha, x_3, x_3^2 + s^2) d\alpha \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_{S^1} s \widehat{k}_\mu(\alpha, \eta, \xi) e^{-i(x_3\eta + (x_3^2 + s^2)\xi)} d\alpha d\eta d\xi \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} s e^{i\frac{\eta^2}{4\xi}} G(\mu, -\frac{\eta}{2\xi}, \xi) e^{-i(x_3\eta + (x_3^2 + s^2)\xi)} d\eta d\xi \end{aligned}$$

Changed the variable η to $2\xi\eta$ gives

$$\sum_{j=1}^2 \int_{S^1} f(\mu + (u + (-1)^j s)\alpha, x_3) d\alpha = \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} s G(\mu, -\eta, \xi) e^{-i(2x_3\eta + (x_3^2 + s^2) - \eta^2)\xi} \xi d\eta d\xi.$$

□

5.2.1 Cylindrical geometry

Let the centers of the central circles be located on the a cylinder $\partial B_R^2(0) \times \mathbb{R} = A \times \mathbb{R}$. That is, A is the circle centered at the origin with radius R . The next two results show that the circular Radon transform can be recovered from the toroidal Radon transform. Both theorems are easily obtained using Lemma 5.2.1.

Theorem 5.2.1. *If $R/2 < u < R$ and $f \in C_c^\infty(B_{R-u}^2(0) \times \mathbb{R})$, then we have*

$$Mf(\mu, x_3, r) = \begin{cases} 2^{-1} I_2^{-1} R_T^* R_T f(\mu, x_3, r - u) & \text{if } r > u, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.2.2. *Let $f \in C_c^\infty(B_R^2(0) \times \mathbb{R})$. Then we have*

$$Mf(\mu, x_3, r) = \begin{cases} \frac{1}{2} \sum_{j=0}^{\lfloor \frac{R}{u} + \frac{1}{2} \rfloor} (-1)^j I_2^{-1} R_T^* R_T f(\mu, x_3, (2j+1)u - r) & \text{if } r \leq u, \\ \frac{1}{2} \sum_{j=0}^{\lfloor R/u \rfloor} (-1)^j I_2^{-1} R_T^* R_T f(\mu, x_3, (2j+1)u + r) & \text{otherwise.} \end{cases}$$

The proofs of Theorems 5.2.1 and 5.2.2 need routine computation.

Remark 5.2.1. *One can obtain other relations similar to Theorems 5.2.1 and 5.2.2, by using Lemma 5.2.2 instead of Lemma 5.2.1.*

Remark 5.2.2. *When the set of centers of the detector circles lie on a cylinder (i.e., A is a circle,) one can recover f from its toroidal transform $R_T f$ by applying inversion*

formulas (see e.g. [23, 48]) to the left hand sides of equations in Theorems 5.2.1 and 5.2.2.

Remark 5.2.3. *If $u > 2R$ (i.e., the radius of central circles is bigger than the diameter of domain cylinder), then*

$$Mf(\mu, x_3, r) = \begin{cases} 2^{-1}I_2^{-1}R_T^*R_Tf(\mu, x_3, u-r) & \text{if } r < u, \\ 2^{-1}I_2^{-1}R_T^*R_Tf(\mu, x_3, u+r) & \text{otherwise.} \end{cases}$$

5.2.2 Planar geometry

Let $A \subset \mathbb{R}^2$ be the line $x_1 = 0$. (i.e., the centers of tori are located on the $x_1 = 0$ plane in \mathbb{R}^3 .) Then $R_Tf(\mu, x_3, r)$ is equal to zero if f is an odd function in x_1 . We thus assume the function f is to be even in x_1 .

Theorem 5.2.3. *Let $f \in C_c^\infty(B_R^3(0))$ be even in x_1 . Then*

$$Mf(\mu, x_3, r) = \begin{cases} \frac{1}{2} \sum_{j=0}^{\lfloor \frac{R+u}{2u} \rfloor} (-1)^j I_2^{-1} R_T^* R_T f(\mu, x_3, (2j+1)u-r) & \text{if } r \leq u, \\ \frac{1}{2} \sum_{j=0}^{\lfloor R/2u \rfloor} (-1)^j I_2^{-1} R_T^* R_T f(\mu, x_3, (2j+1)u+r) & \text{otherwise.} \end{cases} \quad (5.7)$$

Remark 5.2.4. *When A is a line, we can determine f from R_Tf by applying inversion formulas of [7, 21, 55, 57, 61] to the left hand side of (5.7).*

Remark 5.2.5. *If $u > R$ (i.e., the radius of detector is bigger than the radius of the ball containing $\text{supp } f$), then*

$$Mf(\mu, x_3, r) = \begin{cases} 2^{-1}I_2^{-1}R_T^*R_Tf(\mu, x_3, u-r) & \text{if } r < u, \\ 2^{-1}I_2^{-1}R_T^*R_Tf(\mu, x_3, u+r) & \text{otherwise.} \end{cases}$$

6. COMPTON CAMERAS AND CONE TRANSFORMS

Single Photon Emission Computed Tomography (SPECT) is considered a very useful medical diagnostics tool. In particular, it can provide physiological information, while common types of tomography deliver only structural information. A **Compton camera**, also called **electronically collimated γ -camera**, was first introduced in [63, 71] for use in SPECT because of the low efficiency of a conventional gamma camera. The reason for this low efficiency is the use of collimation which reduces significantly the signal-to-noise ratio. Since then, Compton cameras have attracted a lot of interest and applications in many areas including astronomy and monitoring nuclear power plants. Additionally, the Department of Homeland Security has interest in Compton cameras imaging in order to prevent smuggling of weapons-grade nuclear materials.

A typical Compton camera consists of two planar detectors: a scatter detector and an absorption detector, positioned one behind the other. A photon emitted in the direction of the camera undergoes Compton scattering in the scatter detector positioned ahead, and is absorbed in the absorption detector (see Figure 6.1(a)). In

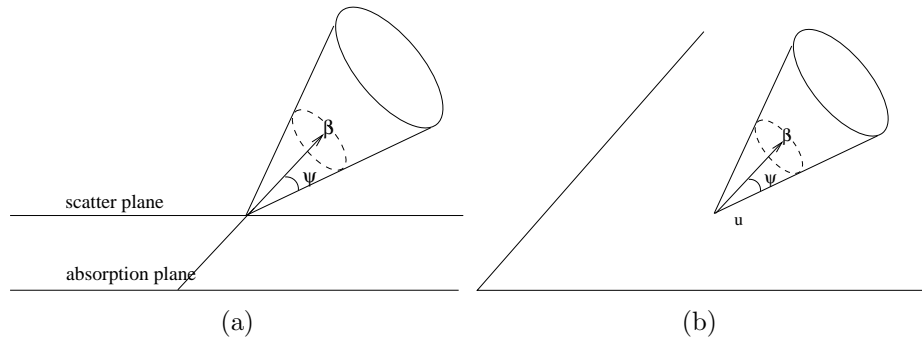


Figure 6.1: Schematic representation of a Compton camera

each detector, the position of the hit and energy of the photon are measured. The scattering angle ψ can be computed by

$$\cos \psi = 1 - \frac{mc^2 \Delta E}{(E - \Delta E)E},$$

where m is the mass of the electron, c is the speed of light, E is the initial gamma-ray energy, and ΔE is the energy transferred to the electron in the scattering process [1, 50]. Also, the direction β is given by difference vector between two device positions. For convenience, we assume β is normalized, i.e., an unit vector. Therefore, we get the integral of the distribution of the radiation source over cone with a central axis β , a vertex u at the position of a scatter detector and scattering angle ψ . We thus reserve the name **cone transform** for the surface integral of a source distribution over this family of cones.

This section is organized as follows. Definition of the cone transform and previous results are introduced in subsection 6.1. In subsection 6.2, we present an inversion formula using complete Compton data for 3 and 2 dimensional cases. Subsection 6.3 is devoted to properties of the cone transform with a fixed central axis.

6.1 The cone transform

Let $f(x, y, z)$ be the distribution of radioactivity sources and let the scatter detectors u be located in xy -plane. We assume that $f(x, y, z)$ is supported on one side of the Compton camera, which means that support of f is contained in the half space $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$. As before, let $\beta \in S^2$ be a central axis and $\psi \in [0, \pi]$ be an opening angle.

The data measured by a Compton camera is the integral of the source distribution over cones which are parameterized by a position vector u , a central axis β , and an

opening angle ψ . Hence, Compton data can be represented as

$$Cf(u, \beta, \psi) = K(\psi) \int_{\beta \cdot \alpha = \cos \psi, \alpha \in S^2} \int_0^\infty f((u, 0) + r\alpha(\phi)) r \sin \psi \, dr d\phi,$$

where $K(\psi)$ is the (known) Klein-Nishina coefficient. Since this factor is known, it can be easily handled, and we do not take it into account in the further analysis.

One considers a more general form of a cone transform $C^k f$, which is

$$C^k f(u, \beta, \psi) = \int_{\beta \cdot \alpha = \cos \psi, \alpha \in S^2} \int_0^\infty f((u, 0) + r\alpha(\phi)) r^k \sin \psi \, dr d\phi,$$

where $k \in \mathbb{R}$. Notice that $C^1 f = Cf$.

There are some previous works containing inversion formulas of this cone transform. While f has 3-dimensional domain, the cone transform $C^k f(u, \beta, \psi)$ depends on 5 variables, so the problem of inverting the cone transform $C^k f(u, \beta, \psi)$ is two dimensions overdetermined. There are many ways of reducing the dimensions of the data, e.g., by fixing a central axis [17, 56, 74] or using 1-dimensional detectors [8, 65]. Some works use the complete set of data for the reconstruction (e.g. [50]). In 2-dimensional case, the problem of inverting the cone transform $C^k f(u, \beta, \psi)$ also is one dimension overdetermined. Various inversions of the 2-dimensional cone transform from the complete set of data are introduced in [1]. All existed works including above works and [37, 59, 72] deal with the cone transform when $k = 0$ or 1 .

6.2 Reconstruction

Analogously to [8], we use spherical harmonics to get an inversion formula. However, while the relation between the cone transform and the Radon transform was the focus in the work [8], the relation between the cone transform and the weighted

fan beam transform is used in this subsection. Our inversion also uses full data, but we provide different inverse formulas than the ones in [1, 8, 50].

6.2.1 3-dimensional case

It is easily verified that the cone transform is shift-invariant, i.e.,

$$C^k(T_a f)(u, \beta, \psi) = T_a C^k f(u, \beta, \psi)$$

for $a = (a_1, a_2) \in \mathbb{R}^2$, where $T_a f(x, y, z) = f(x + a_1, y + a_2, z)$ and $T_a C^k f(u, \beta, \psi) = C^k f(u + a, \beta, \psi)$.

Let $T_u f(\alpha, \rho)$ be a function $T_n f(x, y, z)$ in spherical coordinate. Let us express $(T_u f)(\alpha, \rho)$ and $g^k(u, \beta, \psi) := C^k f(u, \beta, \psi)$ in terms of an expansion in spherical harmonics as follows:

$$(T_u f)(\alpha, \rho) = \sum_{l=0}^{\infty} \sum_{n=0}^{2l+1} (T_u f)_{ln}(\rho) Y_{ln}(\alpha), \quad g^k(u, \beta, \psi) = \sum_{l=0}^{\infty} \sum_{n=0}^{2l+1} g_{ln}^k(u, \psi) Y_{ln}(\beta).$$

Here $Y_{ln}(\omega)$ for $\omega \in S^2$ and $n = 0, 1, 2, \dots, 2l+1, l = 0, 1, \dots$, are spherical harmonics and the coefficients are

$$(T_u f)_{ln}(\rho) = \frac{(2l+1)(l-|n|)!}{4\pi(l+|n|)!} \int_{S^2} T_u f(\alpha, \rho) Y_{ln}(\alpha) dS(\alpha),$$

$$g_{ln}^k(u, \psi) = \frac{(2l+1)(l-|n|)!}{4\pi(l+|n|)!} \int_{S^2} g^k(u, \beta, \psi) Y_{ln}(\beta) dS(\beta),$$

where dS is the surface measure on the unit sphere S^2 in \mathbb{R}^3 . The following theorem describes a relation between two coefficients $(T_u f)_{ln}$ and g_{ln}^k .

Theorem 6.2.1. *Let $f \in \mathcal{S}(\mathbb{R}^3)$. Then we have*

$$\int_0^\infty (T_u f)_{ln}(\rho) \rho^k d\rho = \frac{(2l+1)}{4\pi} \int_0^\pi g_{ln}^k(u, \psi) P_l(\cos \psi) d\psi,$$

where $P_l(t)$ is the Legendre polynomial of degree l .

Proof. By definition, we have

$$\begin{aligned} C^k(T_u f)(0, \beta, \psi) &= \int_0^\infty \int_{\alpha \cdot \beta = \cos \psi} (T_u f)(\alpha, \rho) \rho^k \sin \psi d\rho d\alpha \\ &= \sum_{l=0}^\infty \sum_{n=0}^{2l+1} \int_0^\infty \int_{\alpha \cdot \beta = \cos \psi} (T_u f)_{ln}(\rho) Y_{ln}(\alpha) \rho^k \sin \psi d\rho d\alpha \\ &= \sum_{l=0}^\infty \sum_{n=0}^{2l+1} \int_0^\infty (T_u f)_{ln}(\rho) \rho^k \sin \psi d\rho \int_{\alpha \cdot \beta = \cos \psi} Y_{ln}(\alpha) d\alpha. \end{aligned} \quad (6.1)$$

The Funk-Hecke theorem [54, 62] is

$$\int_{S^2} h(\theta \cdot \omega) Y_{ln}(\omega) d\omega = c(l) Y_{ln}(\theta), \quad c(l) = 2\pi \int_{-1}^1 h(t) P_l(t) dt,$$

for $\theta \in S^2$ and a function h is on $[-1, 1]$. Using this theorem, we get

$$\int_{\cos \psi = \alpha \cdot \beta} Y_{ln}(\alpha) d\alpha = \int_{\cos \psi = t, -1 \leq t \leq 1} P_l(t) dt Y_{ln}(\beta) = 2\pi P_l(\cos \psi) Y_{ln}(\beta).$$

Substituting the above equation into (6.1), we get

$$\begin{aligned} g^k(u, \beta, \psi) &= T_u C^k f(0, \beta, \psi) = C^k(T_u f)(0, \beta, \psi) \\ &= 2\pi \sum_{l=0}^\infty \sum_{n=0}^{2l+1} P_l(\cos \psi) \sin \psi \int_0^\infty (T_u f)_{ln}(\rho) \rho^k d\rho Y_{ln}(\beta), \end{aligned}$$

which implies

$$g_{ln}^k(u, \psi) = 2\pi P_l(\cos \psi) \sin \psi \int_0^\infty (T_u f)_{ln}(\rho) \rho^k d\rho. \quad (6.2)$$

The known relation $\int_0^\pi |P_l(\cos \psi)|^2 \sin \psi d\psi = 2/(2l+1)$ yields the assertion. \square

Remark 6.2.1. *Relation (6.2) was mentioned in [8] for $k = 1$ in a different manner.*

Remark 6.2.2. *When we need to use the Klein-Nishina formula, we can multiply (6.2) by $K(\psi)$ and integrate it with respect to ψ . Since $\int_0^\pi |P_l(\cos \psi)K(\psi)|^2 \sin \psi d\psi$ is positive, we have*

$$\int_0^\infty (T_u f)_{ln}(\rho) \rho^k d\rho = \frac{1}{2\pi} \left(\int_0^\pi |P_l(\cos \psi)K(\psi)|^2 \sin \psi d\psi \right)^{-1} \int_0^\pi g_{ln}^k(u, \psi) P_l(\cos \psi) d\psi.$$

Theorem 6.2.2. *Let $f \in \mathcal{S}(\mathbb{R}^3)$. Then*

$$D^k f(u, \alpha) := \int_0^\infty f((u, 0) + r\alpha) r^k dr, \quad \text{for } (u, \alpha) \in \mathbb{R}^2 \times S^2$$

can be obtained through

$$\sum_{l=0}^\infty \sum_{n=0}^{2l+1} \frac{(2l+1)}{4\pi} \int_0^\pi g_{ln}^k(u, \psi) P_l(\cos \psi) d\psi Y_{ln}(\alpha).$$

Remark 6.2.3. *When k is equal to zero, we obtain a relation between the fan beam transform and the cone transform:*

$$D^0 f(u, \alpha) = \sum_{l=0}^\infty \sum_{n=0}^{2l+1} \frac{(2l+1)}{4\pi} \int_0^\pi g_{ln}^0(u, \psi) P_l(\cos \psi) d\psi Y_{ln}(\alpha).$$

In fact, we have not used the condition that u is located on the xy -plane. Hence we can apply the theory of the fan beam transform. In particular, Finch [22] showed when

$f \in C_0^2(U)$, where $U \subset \mathbb{R}^3$ is open and bounded, and C is a connected curve lying in $\mathbb{R}^3 \setminus \bar{U}$, f can be determined from $D^0 f(u, \alpha)$, i.e., $C^0 f(u, \beta, \psi)$, if $E_\theta(\text{supp} f) \subset E_\theta(\text{convex hull } C)$ for each $\theta \in S^2$ in a nonempty open cone Γ where E_θ denotes orthogonal projection on θ^\perp . In other words, if the position u of a scatter detector is located on a curve C satisfying $E_\theta(\text{supp} f) \subset E_\theta(\text{convex hull } C)$, we can find f from $C^0 f(u, \beta, \psi)$.

To find an inversion formula for the cone transform $C^k f$, we need an inversion formula of $D^k f(u, \alpha)$ where $u \in \mathbb{R}^2$.

Theorem 6.2.3. *For any $k \in \mathbb{R}$ and for $f(x, y, z) \in \mathcal{S}(\mathbb{R}^3)$ compactly supported in an upper half plane $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, the following formula holds:*

$$\hat{f}(\xi, z) = -z^{1-k} \int_0^\infty \int_{-\pi}^\pi \mathcal{F}(\Delta_u D^k f)(\xi, \theta, \arctan s) J_0(zs|\xi|) \frac{s}{(1+s^2)^{(k+1)/2}} d\theta ds,$$

where Δ_u is the Laplace operator in the variable u and \hat{f} and $\mathcal{F}(D^k f)$ (or $\widehat{D^k f}$) are the Fourier transforms of f and $D^k f$ with respect to the 2-dimensional variables (x, y) and u , respectively.

Proof. Let θ and ϕ be the azimuthal and polar angles of α , respectively. Taking the Fourier transform of $D^k f(u, \alpha)$ with respect to u , we obtain

$$\widehat{D^k f}(\xi, \theta, \phi) = \int_0^\infty \hat{f}(\xi, r \cos \phi) e^{ir\xi \cdot (\cos \theta, \sin \theta) \sin \phi} r^k dr,$$

where \hat{f} is the 2-dimensional Fourier transform of f with respect to the first two

variables (x, y) . Integrating $\widehat{D^k f}(\xi, \theta, \phi)$ with respect to θ gives

$$\begin{aligned} \int_{-\pi}^{\pi} \widehat{D^k f}(\xi, \theta, \phi) d\theta &= \int_0^{\infty} \hat{f}(\xi, r \cos \phi) r^k \int_{-\pi}^{\pi} e^{ir\xi \cdot (\cos \theta, \sin \theta) \sin \phi} d\theta dr \\ &= \int_0^{\infty} \hat{f}(\xi, r \cos \phi) r^k \int_{-\pi}^{\pi} e^{ir|\xi| \cos(\theta + \varphi) \sin \phi} d\theta dr \\ &= \int_0^{\infty} \hat{f}(\xi, r) r^k J_0(r \tan \phi |\xi|) dr \cos^{-k-1} \phi, \end{aligned}$$

for a fixed $\xi = |\xi|(\cos \varphi, \sin \varphi)$. In the last line, we changed variables $r \rightarrow r/\cos \phi$. If we consider $\hat{f}(\xi, r)$ as a function of the variable r for a fixed ξ , then the above integral is the Hankel transform of zero order of $\hat{f}(\xi, r)r^{k-1}$. Hence applying the Hankel transform of zero order and changing variables, we get

$$\begin{aligned} \hat{f}(\xi, r) &= r^{1-k} \int_0^{\infty} \int_{-\pi}^{\pi} \widehat{D^k f}(\xi, \theta, \phi) J_0(r \tan \phi |\xi|) |\xi|^2 \tan \phi \cos^{k+1} \phi d\theta (d \tan \phi) \\ &= -r^{1-k} \int_0^{\infty} \int_{-\pi}^{\pi} \mathcal{F}(\Delta_u D^k f)(\xi, \theta, \arctan s) J_0(rs|\xi|) \frac{s}{(1+s^2)^{(k+1)/2}} d\theta ds. \end{aligned}$$

□

Therefore, we obtain the following inversion formula by combining Theorems 6.2.2 and 6.2.3.

Theorem 6.2.4. *If $f(x, y, z) \in \mathcal{S}(\mathbb{R}^3)$ has compact support in $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, then f can be obtained from $C^k f = g^k$ as follows:*

$$\sum_{l=0}^{\infty} \frac{2(2l+1)\pi^2}{|u|} \int_{\mathbb{R}^2} \int_0^{\pi} \Delta_u g_{l0}^k((x, y) - u, \psi) \frac{P_l(\cos \psi)}{(z^2 + |u|^2)^{\frac{k+1}{2}}} P_l \left(\frac{z}{\sqrt{z^2 + |u|^2}} \right) d\psi du.$$

Proof. We know $Y_{ln}(\alpha) = P_l^{|n|}(\cos \phi)e^{-in\theta}$, where $P_l^n(x)$ is the associated Legendre function. Hence, we can write $\hat{f}(\xi, z)$ as

$$-\sum_{l=0}^{\infty} \sum_{n=0}^{2l+1} \frac{(2l+1)z^{1-k}}{4\pi} \int_0^{\infty} \int_{-\pi}^{\pi} \int_0^{\pi} \left\{ \begin{aligned} &\mathcal{F}(\Delta_u g_{ln}^k)(\xi, \psi) P_l(\cos \psi) e^{-in\theta} \\ &\times P_l^{|n|} \left(\frac{1}{\sqrt{1+s^2}} \right) J_0(zs|\xi|) \frac{s}{(1+s^2)^{(k+1)/2}} \end{aligned} \right\} d\psi d\theta ds.$$

Here summation commutes with integration because the integrand is absolutely integrable with respect to s and ψ . Since

$$\int_{-\pi}^{\pi} e^{-in\theta} d\theta = \begin{cases} 2\pi & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we have that $f(x, y, z)$ is equal to

$$-\sum_{l=0}^{\infty} \frac{(2l+1)z^{1-k}}{2} \int_{\mathbb{R}^2} \int_0^{\pi} \int_0^{\infty} \left\{ \begin{aligned} &\mathcal{F}(\Delta_u g_{l0}^k)(\xi, \psi) \frac{s}{(1+s^2)^{\frac{k+1}{2}}} \\ &\times P_l(\cos \psi) P_l \left(\frac{1}{\sqrt{1+s^2}} \right) J_0(zs|\xi|) e^{i\xi \cdot (x,y)} \end{aligned} \right\} ds d\psi d\xi.$$

Consider the inner integral with respect to $d\xi$:

$$\int_{\mathbb{R}^2} \mathcal{F}(\Delta_u g_{l0}^k)(\xi, \psi) J_0(zs|\xi|) e^{i\xi \cdot (x,y)} d\xi.$$

Since the Fourier transform of $J_0(zs|\xi|)$ in ξ is $2\pi\delta(|(x,y)| - zs)(zs)^{-1}$, where δ is the Dirac delta function, we have

$$\int_{\mathbb{R}^2} \mathcal{F}(\Delta_u g_{l0}^k)(\xi, \psi) J_0(zs|\xi|) e^{i\xi \cdot (x,y)} d\xi = (2\pi)^3 \int_{\mathbb{R}^2} \Delta_u g_{l0}^k((x,y) - u, \psi) \delta(|u| - zs) \frac{du}{zs}.$$

□

6.2.2 2-dimensional case

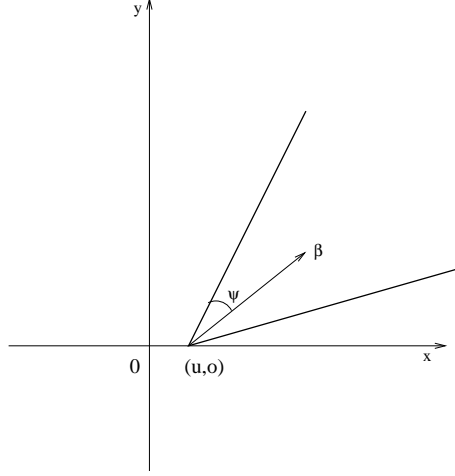


Figure 6.2: A sketch of a V -line transform

In two dimensions, the cone transform becomes the integral of f over a “broken line” (or a V -line) (see Figure 6.2). These V -lines have a common vertex u and two directions $(\cos(\beta \pm \psi), \sin(\beta \pm \psi))$. In this case, the scatter position becomes 1-dimensional and is assumed to be located on the x -axis. Hence, one consider a V -line transform of the form

$$\begin{aligned} & V^k f(u, \beta, \psi) \\ &= \int_0^\infty [f((u, 0) + r(\cos(\beta - \psi), \sin(\beta - \psi))) - f((u, 0) + r(\cos(\beta + \psi), \sin(\beta + \psi)))] r^k dr, \end{aligned}$$

where $\beta \in [0, 2\pi)$. It is also clear that our V -line transform has shift invariance with respect to $u \in \mathbb{R}$.

Let us consider the Fourier series of $(T_u f)(\alpha, \rho)$ and $g^k(u, \beta, \psi) := V^k f(u, \beta, \psi)$

with respect to the angular variables α and β . Then we get

$$(T_u f)(\alpha, \rho) = \sum_{n=-\infty}^{\infty} (T_u f)_n(\rho) e^{in\alpha}, \quad g^k(u, \beta, \psi) = \sum_{n=-\infty}^{\infty} g_n^k(u, \psi) e^{in\beta}$$

where the Fourier coefficients are

$$(T_u f)_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} T_u f(\alpha, \rho) e^{in\alpha} d\alpha \quad \text{and} \quad g_n^k(u) = \frac{1}{2\pi} \int_0^{2\pi} g^k(u, \beta) e^{in\beta} d\beta.$$

A relation between two Fourier coefficients $(T_u f)_n$ and g_n^k can be described as follows:

Theorem 6.2.5. *For $f \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\int_0^{\infty} (T_u f)_n(\rho) \rho^k d\rho = \frac{1}{\pi} \int_0^{\pi} g_n^k(u, \psi) \cos(n\psi) d\psi.$$

Proof. We can write $V^k(T_u f)(0, \beta, \psi)$ as

$$\begin{aligned} & \int_0^{\infty} [(T_u f)(\beta - \psi, \rho) + (T_u f)(\beta + \psi, \rho)] \rho^k d\rho \\ &= \sum_{n=-\infty}^{\infty} \int_0^{\infty} (T_u f)_n(\rho) (e^{in(\beta-\psi)} + e^{in(\beta+\psi)}) \rho^k d\rho \\ &= 2 \sum_{n=-\infty}^{\infty} \int_0^{\infty} (T_u f)_n(\rho) \rho^k d\rho \cos(n\psi) e^{in\beta}. \end{aligned}$$

Therefore, we get

$$g^k(u, \beta, \psi) = T_u g^k(0, \beta, \psi) = V^k(T_u f)(0, \beta, \psi) = 2 \sum_{n=-\infty}^{\infty} \int_0^{\infty} (T_u f)_n(\rho) \rho^k d\rho \cos(n\psi) e^{in\beta},$$

which implies our assertion, using the identity

$$\int_0^\pi \cos^2(n\psi) d\psi = \frac{\pi}{2}.$$

□

Theorem 6.2.6. *Let $D_2^k f(u, \alpha)$ for $f \in \mathcal{S}(\mathbb{R}^2)$ be defined as*

$$D_2^k f(u, \alpha) := \int_0^\infty f((u, 0) + r(\cos \alpha, \sin \alpha)) r^k dr, \quad \text{for } (u, \alpha) \in \mathbb{R} \times [0, 2\pi).$$

Then

$$D_2^k f(u, \alpha) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \int_0^\pi g_n^k(u, \psi) \cos(n\psi) d\psi e^{in\alpha}.$$

As in the 3-dimensional case, an inversion formula of D_2^k is required.

Theorem 6.2.7. *Let $f(x, y) \in \mathcal{S}(\mathbb{R}^2)$ have support in $\{(x, y) \in \mathbb{R}^2 : y > 0\}$. Then*

$$f(x, y) = -\sqrt{2\pi} i y^{-k} \int_0^\pi \frac{\partial}{\partial u} D_2^k f(x - y \cot \alpha, \alpha) \sin^{k-1} \alpha d\alpha.$$

Proof. Taking the Fourier transform of $D_2^k f(u, \beta)$ with respect to u , we obtain

$$\widehat{D_2^k f}(\xi, \alpha) = \int_0^\infty \hat{f}(\xi, r \sin \alpha) e^{ir \cos \alpha \xi} r^k dr = \frac{1}{\sin^{k+1} \alpha} \int_0^\infty \hat{f}(\xi, r) e^{ir \xi \cot \alpha} r^k dr,$$

where \hat{f} is the Fourier transform of f with respect to the first variable x . Here we changed the variable r to $r/\sin \alpha$. Multiplying $e^{-ia\xi \cot \alpha} \sin^{k-1} \alpha$ by $\widehat{D_2^k f}(\xi, \alpha)$ for

any $a > 0$ and integrating with respect to θ give

$$\begin{aligned}
\int_0^\pi \widehat{D_2^k f}(\xi, \alpha) e^{-ia\xi \cot \alpha} \sin^{k-1} \alpha d\alpha &= \int_0^\pi \int_0^\infty \hat{f}(\xi, r) e^{ir\xi \cot \alpha - ia\xi \cot \alpha} r^k dr \sin^{-2} \alpha d\alpha \\
&= \int_{\mathbb{R}} \int_0^\infty \hat{f}(\xi, r) e^{i(r-a)\xi t} r^k dr dt \\
&= \sqrt{2\pi} \int_0^\infty \hat{f}(\xi, r) \delta((r-a)\xi) r^k dr \\
&= \sqrt{2\pi} \xi^{-1} \hat{f}(\xi, a) a^k,
\end{aligned}$$

where in the second line, we changed variables $\cot \alpha \rightarrow t$. Thus, we have

$$\hat{f}(\xi, y) = -\sqrt{2\pi} y^{-k} \int_0^\pi i(i\xi) \widehat{D_2^k f}(\xi, \alpha) e^{-iy\xi \cot \alpha} \sin^{k-1} \alpha d\alpha.$$

□

Combining Theorems 6.2.6 and 6.2.7, we get the following theorem:

Theorem 6.2.8. *Let $f(x, y) \in \mathcal{S}(\mathbb{R}^2)$ have support in $\{(x, y) \in \mathbb{R}^2 : y > 0\}$. Then*

$$f(x, y) = -i\sqrt{\frac{2}{\pi}} \sum_{n=-\infty}^{\infty} \int_0^\pi \int_0^\pi y^{-k} \frac{\partial}{\partial u} g_n^k(x - y \cot \alpha, \psi) \cos(n\psi) e^{in\alpha} \sin^{k-1} \alpha d\psi d\alpha.$$

Remark 6.2.4. *Similarly to 3-dimensional case, when k is equal to zero, we obtain a relation between the fan beam transform and a V-line transform:*

$$D_2^0 f(u, \alpha) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \int_0^\pi g_n^0(u, \psi) \cos(n\psi) d\psi e^{in\alpha}. \quad (6.3)$$

To get (6.3), we does not need the condition that the detector position variable u is located on the x -axis. We thus can apply any inversion formulas for the fan beam

transform.

When $f \in C_0^2(U)$, where $U \subset \mathbb{R}^2$ is open and bounded, and C is a connected curve lying in $\mathbb{R}^2 \setminus \bar{U}$, f can be determined from $D_2^0 f(u, \alpha)$, i.e., $V^0 f(u, \beta, \psi)$, if $E_\theta(\text{supp} f) \subset E_\theta(\text{convexhull} C)$ for each θ in a nonempty open cone Γ where E_θ denotes orthogonal projection on θ^\perp .

6.3 The cone transform with a fixed central axis

As we already know, the inverting problem of $C^k f$ is two dimensions overdetermined. Like in [17, 56], we will fix a central axis β parallel to z -axis. Then the (reduced) cone transform becomes

$$C^k f(u, (0, 0, 1), \psi) = \int_0^\infty \int_{-\pi}^\pi f(u_1 + z \tan \psi \cos \theta, u_2 + z \tan \psi \sin \theta, z) z \tan^k \psi d\theta dz.$$

Let us change notation. We let $s = \tan \psi$ and denote

$$C_{fix}^k f(u, s) := s \sqrt{1 + s^2} \int_0^\infty \int_{-\pi}^\pi f(u_1 + zs \cos \theta, u_2 + zs \sin \theta, z) z^k d\theta dz,$$

where $k \in \mathbb{R}$ and $(u, s) \in \mathbb{R}^2 \times [0, \infty)$.

Our goal is to find inversion formulas for $C_{fix}^k f(u, s)$ and study their properties.

6.3.1 Inversion formulas

We have an analogue of the Fourier slice theorem.

Theorem 6.3.1. *Let $f \in C_c^\infty(\mathbb{R}^2 \times (0, \infty))$. Then we have*

$$\hat{f}(\xi, z) = \frac{z^{1-k}}{2\pi} \int_0^\infty \frac{\widehat{C_{fix}^k f}(\xi, s)}{\sqrt{1 + s^2}} J_0(zs|\xi|) |\xi|^2 ds.$$

Proof. It is easy to check that the cone transform with a fixed central axis is shift-invariant: $C_{fix}^k(T_{(a,b)}f)(u, s) = C_{fix}^k f(u_1+a, u_2+b, s)$ for $a, b \in \mathbb{R}$, where $T_{(a,b)}f(x, y, z) = f(x+a, y+b, z)$. It, thus, is reasonable to take the Fourier transform with respect to u . Doing this, we obtain

$$\begin{aligned}\widehat{C_{fix}^k f}(\xi, s) &= s\sqrt{1+s^2} \int_0^\infty \int_{-\pi}^\pi \hat{f}(\xi, z) e^{izs\xi \cdot (\cos\theta, \sin\theta)} z^k d\theta dz \\ &= s\sqrt{1+s^2} \int_0^\infty \hat{f}(\xi, z) \int_{-\pi}^\pi e^{izs|\xi| \cos(\theta-\phi)} z^k d\theta dz \\ &= 2\pi s\sqrt{1+s^2} \int_0^\infty \hat{f}(\xi, z) J_0(zs|\xi|) z^k dz,\end{aligned}\tag{6.4}$$

where $\xi = |\xi|(\cos\phi, \sin\phi)$ and \hat{f} and $\widehat{C_{fix}^k f}$ are the 2-dimensional Fourier transforms of f and $C_{fix}^k f$ with respect to (x, y) and u , correspondingly. For a fixed ξ , we can think of $\hat{f}(\xi, z)$ as a one variable function of z . The last integral in (6.4) is, up to the factor $s\sqrt{1+s^2}$, the Hankel transform of order zero of $z^{k-1}\hat{f}(\xi, z)$. Since $f(x, y, z)$ has support in $\{(x, y, z) : z > 0\}$, applying the Hankel transform yields our assertion. \square

Theorem 6.3.2. *Let $f \in C_c^\infty(\mathbb{R}^2 \times (0, \infty))$. Then we have for $z > 0$,*

$$f(x, y, z) = -\frac{z^{1-k}}{2\pi} \int_{\mathbb{R}^2} \frac{\Delta_u C_{fix}^k f(x', y', |(x-x', y-y')|/z)}{|(x-x', y-y')| \sqrt{|(x-x', y-y')|^2 + z^2}} dx' dy',$$

where Δ_u is the Laplace operator with respect to the variable u .

Proof. From Theorem 6.3.1, we have

$$\hat{f}(\xi, z) = -\frac{z^{1-k}}{2\pi} \int_0^\infty \mathcal{F}(\Delta_u C_{fix}^k f)(\xi, s) J_0(zs|\xi|) \frac{ds}{\sqrt{1+s^2}},\tag{6.5}$$

where $\mathcal{F}(C_{fix}^k f)$ is the Fourier transform of $C_{fix}^k f$. Hence we have

$$f(x, y, z) = -\frac{1}{(2\pi)^3} z^{1-k} \int_{\mathbb{R}^2} \int_0^\infty \mathcal{F}(\Delta_u C_{fix}^k f)(\xi, s) J_0(zs|\xi|) \frac{ds}{\sqrt{1+s^2}} e^{i\xi \cdot (x,y)} d\xi. \quad (6.6)$$

Consider $\int_{\mathbb{R}^2} J_0(zs|\xi|) e^{i\xi \cdot (x,y)} d\xi$. Switching the Cartesian coordinate ξ to polar coordinates (ρ, ϕ) yields

$$\begin{aligned} \int_{\mathbb{R}^2} J_0(zs|\xi|) e^{i\xi \cdot (x,y)} d\xi &= \int_{-\pi}^{\pi} \int_0^\infty J_0(zs\rho) e^{i\rho(\cos\phi, \sin\phi) \cdot (x,y)} \rho d\rho d\phi \\ &= \int_0^\infty J_0(zs\rho) \rho \int_{-\pi}^{\pi} e^{i\rho|(x,y)|\cos(\phi-\theta)} d\phi d\rho \\ &= 2\pi \int_0^\infty J_0(zs\rho) J_0(\rho|(x,y)|) \rho d\rho = 2\pi(zs)^{-1} \delta(zs - |(x,y)|), \end{aligned}$$

where $(x, y) = |(x, y)|(\cos\theta, \sin\theta)$. Hence, we can simplify (6.6) as follows:

$$\begin{aligned} f(x, y, z) &= -\frac{1}{(2\pi)^3} z^{1-k} \int_{\mathbb{R}^2} \int_0^\infty \widehat{\Delta_u C_{fix}^k f}(\xi, s) J_0(zs|\xi|) \frac{ds}{\sqrt{1+s^2}} e^{i\xi \cdot (x,y)} d\xi \\ &= -\frac{z^{1-k}}{2\pi} \int_{\mathbb{R}^2} \int_0^\infty \Delta_u C_{fix}^k f(x', y', s) (zs)^{-1} \delta(zs - |(x-x', y-y')|) \frac{ds}{\sqrt{1+s^2}} dx' dy' \\ &= -\frac{z^{1-k}}{2\pi} \int_{\mathbb{R}^2} \frac{\Delta_u C_{fix}^k f(x', y', |(x-x', y-y')|/z)}{|(x-x', y-y')| \sqrt{|(x-x', y-y')|^2 + z^2}} dx' dy'. \end{aligned}$$

□

Remark 6.3.1. When $k = 1$, (6.6) was obtained in [17].

Let $\hat{f}(\varrho, \varphi, z)$ and $\hat{g}^k(\rho, \phi, s)$ be the functions $\hat{f}(\xi, z)$ and $\widehat{C_{fix}^k f}(\eta, s)$ in the polar coordinates $(\varrho, \varphi) \in [0, \infty) \times S^1$ and $(\rho, \phi) \in [0, \infty) \times S^1$, where $\varrho = |\xi|, \varphi = \xi/|\xi|$ and $\rho = |\eta|, \phi = \eta/|\eta|$. Then the Fourier series of $\hat{f}(\varrho, \varphi, z)$ and $\hat{g}^k(\rho, \phi, s)$ and with

respect to the polar angles φ and ϕ can be written as follows:

$$\hat{f}(\varrho, \varphi, z) = \sum_{n=-\infty}^{\infty} F f_n(\varrho, z) e^{in\varphi}, \quad \hat{g}^k(\rho, \phi, s) = \sum_{n=-\infty}^{\infty} F g_n^k(\rho, s) e^{in\phi},$$

where

$$F f_n(\varrho, z) = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}(\varrho, \varphi, z) e^{-in\varphi} d\varphi, \quad F g_n^k(\rho, s) = \frac{1}{2\pi} \int_0^{2\pi} \hat{g}^k(\rho, \phi) e^{-in\phi} d\phi.$$

The following theorem describes the relation between $F g_n^k$ and $F f_n$.

Theorem 6.3.3. *For $f \in C_c^\infty(\mathbb{R}^2 \times (0, \infty))$, we have*

$$F g_n^k(\rho, s) = 2\pi s \sqrt{1+s^2} \int_0^\infty F f_n(\rho, z) J_0(zs\rho) z^k dz. \quad (6.7)$$

Proof. From (6.4), we have

$$\begin{aligned} F g_n^k(\rho, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{g}^k(\rho, \phi) e^{-in\phi} d\phi = s \sqrt{1+s^2} \int_0^{2\pi} \int_0^\infty \hat{f}(\rho, \phi, z) J_0(zs\rho) z^k e^{-in\phi} dz d\phi \\ &= 2\pi s \sqrt{1+s^2} \int_0^\infty F f_n(\rho, z) J_0(zs\rho) z^k dz. \end{aligned}$$

□

The right hand side of (6.7) can be viewed as the Hankel transform of zero order of $F f_n(\rho, z)$. Hence we have the following corollary.

Corollary 6.3.1. *For $f \in C_c^\infty(\mathbb{R}^2 \times (0, \infty))$, we have*

$$F f_n(\varrho, z) = \frac{\varrho^2}{2\pi z^{k-1}} \int_0^\infty F g_n^k(\varrho, s) J_0(s\varrho z) \frac{ds}{\sqrt{1+s^2}}.$$

Consider now $Fg_n^k(\rho, s)$:

$$\begin{aligned}
Fg_n^k(\rho, s) &= \frac{1}{2\pi} \int_0^{2\pi} \widehat{g}^k(\rho, \phi, s) e^{-in\phi} d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty g^k(r, \theta, s) e^{-ir\rho(\cos\theta, \sin\theta) \cdot (\cos\phi, \sin\phi) - in\phi} r dr d\theta d\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} g^k(r, \theta, s) e^{-ir\rho \cos(\theta-\phi) - in\phi} r d\phi dr d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} g^k(r, \theta, s) e^{-ir\rho \cos\phi - in(\phi+\theta)} r d\phi dr d\theta,
\end{aligned}$$

where in the third line, we used the Fubini-Tonelli theorem and in the last line, we changed the variable $\phi - \theta$ to ϕ . Let g_n^k be the n -th Fourier coefficient of g^k with respect to the polar angle. Continuing the computation of Fg_n^k gives

$$Fg_n^k(\rho, s) = \int_0^{2\pi} \int_0^\infty g_n^k(r, s) e^{-ir\rho \cos\phi - in\phi} r dr d\phi = 2\pi(-i)^n \int_0^\infty g_n^k(r, s) J_n(r\rho) r dr. \quad (6.8)$$

Remark 6.3.2. When $k = 1$, Theorem 6.3.3 and Corollary 6.3.1 are already obtained in [56]. Combining Corollary 6.3.1 and (6.8), Nguyen et al. obtained

$$\hat{f}(\varrho, \phi, z) = \varrho^2 \int_0^\infty \int_0^\infty G(\varrho, \rho, \phi, s) J_0(s\varrho z) \frac{\rho ds d\rho}{\sqrt{1+s^2}},$$

where

$$G(\varrho, \rho, \phi, s) = \sum_{n=-\infty}^\infty i^n g_n^1(\rho, s) J_n(\rho\varrho) e^{in\phi}.$$

Corollary 6.3.2. *For $f \in C_c^\infty(\mathbb{R}^2 \times (0, \infty))$, we have*

$$f_n(r, z) = \frac{1}{2\pi z^{k-1}} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^3}{\sqrt{1+s^2}} \nu g_n^k(\nu, s) J_n(\rho\nu) J_0(s\rho z) J_n(r\rho) d\nu ds d\rho.$$

Proof. Similarly to (6.8), we get

$$F f_n(r, z) = 2\pi(-i)^n \int_0^\infty f_n(\rho, z) J_n(r\rho) \rho d\rho. \quad (6.9)$$

Note that the right hand side of (6.9) is the Hankel transform of n -th order of f_n , so we have

$$f_n(r, z) = \frac{i^n}{2\pi} \int_0^\infty F f_n(\rho, z) J_n(r\rho) \rho d\rho.$$

Using Corollary 6.3.1, we obtain the following equation:

$$\begin{aligned} f_n(r, z) &= \frac{i^n}{4\pi^2} \int_0^\infty \frac{\rho^2}{z^{k-1}} \int_0^\infty F g_n^k(\rho, s) J_0(s\rho z) J_n(r\rho) \rho \frac{ds}{\sqrt{1+s^2}} d\rho \\ &= \frac{1}{2\pi z^{k-1}} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\rho^3}{\sqrt{1+s^2}} \nu g_n^k(\nu, s) J_n(\rho\nu) J_0(s\rho z) J_n(r\rho) d\nu ds d\rho. \end{aligned}$$

□

6.3.2 A stability estimate

Let $\gamma \geq 0$ and let $L_{(s+s^3)^{-1}}^2(\mathbb{R}^2 \times [0, \infty))$ be the set of a measurable function g with

$$\int_{\mathbb{R}^2} \int_0^\infty |g(u, s)|^2 (s + s^3)^{-1} ds du < \infty.$$

For $f \in C_c^\infty(\mathbb{R}^2 \times (0, \infty))$ and $g \in L_{(s+s^3)^{-1}}^2(\mathbb{R}^2 \times [0, \infty))$ smooth in u , define $\|f\|_\gamma^2$ and $\|g\|_{\gamma+1}^2$ by

$$\int_0^\infty \int_{\mathbb{R}^2} |\hat{f}(\xi, z)|^2 (|\xi|^2 + 1)^\gamma z^{2k-1} d\xi dz \text{ and } \int_{\mathbb{R}^2} \int_0^\infty |\widehat{g}(\xi, s)|^2 (|\xi|^2 + 1)^{\gamma+1} (s + s^3)^{-1} ds d\xi.$$

Theorem 6.3.4. *Let B_1^2 be the unit ball in \mathbb{R}^2 . We have that for each $\gamma \geq 0$ and for $f \in C_c^\infty(B_1^2 \times (0, \infty))$,*

$$\|f\|_\gamma \leq \|C_{fix}^k f\|_{\gamma+1}.$$

Proof. By the Plancherel theorem for the Hankel transform (1.2), we have

$$\int_0^\infty |z^{k-1} \hat{f}(\xi, z)|^2 z dz = \int_0^\infty |H_0(z^{k-1} \hat{f})(\xi, s)|^2 s ds.$$

Hence, we get

$$\begin{aligned} \int_{\mathbb{R}^2} \int_0^\infty |\hat{f}(\xi, z)|^2 z^{2k-1} dz d\xi &= \int_{\mathbb{R}^2} \int_0^\infty |H_0(z^{k-1} \hat{f})(\xi, s)|^2 s ds d\xi \\ &= \int_{\mathbb{R}^2} \int_0^\infty (s + s^3 |\xi|^{-2})^{-1} |\xi|^2 |\widehat{C_{fix}^k f}(\xi, s|\xi|^{-1})|^2 ds d\xi \quad (6.10) \\ &= \int_{\mathbb{R}^2} \int_0^\infty |\xi|^2 |\widehat{C_{fix}^k f}(\xi, s)|^2 (s + s^3)^{-1} ds d\xi. \end{aligned}$$

In the first line, we used the identity $\widehat{C_{fix}^k f}(\xi, s) = H_0(z^{k-1} \hat{f})(\xi, s|\xi|) s \sqrt{1 + s^2}$ or

$$|\xi| (s \sqrt{1 + (s/|\xi|)^2})^{-1} \widehat{C_{fix}^k f}(\xi, s|\xi|^{-1}) = H_0(z^{k-1} \hat{f})(\xi, s)$$

obtained from (6.4). Substituting $(|\xi|^2 + 1)^{\gamma+1}|\xi|^{-2}d\xi$ for $d\xi$ on (6.10) yields

$$\int_{\mathbb{R}^2} \int_0^\infty |\hat{f}(\xi, z)|^2 z^{2k-1} |\xi|^{-2} (|\xi|^2 + 1)^{\gamma+1} dz d\xi \leq \|C_{fix}^k f\|_{\gamma+1}.$$

Therefore, we have $\|f\|_\gamma \leq \|C_{fix}^k f\|_{\gamma+1}$. □

7. INVERSION OF THE WINDOWED RAY TRANSFORM

The windowed ray transform was introduced in [43] by Kaiser and Streater. It is a natural generalization of the “Analytic-Signal Transform” [42] arising from a method for extending arbitrary functions from \mathbb{R}^n to \mathbb{C}^n in a semi-analytic way in relativistic quantum theory. Namely, the Analytic-Signal Transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is the function $g : \mathbb{C}^n \rightarrow \mathbb{C}$ defined by

$$g(u + iv) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(u + \tau v)}{\tau - i} d\tau.$$

Its generalization, the windowed ray transform, is defined as

$$P_h f(u, v) = \int_{\mathbb{R}} f(u + tv) h(t) dt, \text{ for } (u, v) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0.$$

Here h is regarded as a window, which explains the terminology **windowed ray transform**. When $h = 1$ and $\|v\| = 1$, it becomes the usual X-ray transform. In order to minimize analytical subtleties, we assume that h is smooth with rapid decay, i.e., $h \in \mathcal{S}(\mathbb{R})$.

In this section we present several inversion formulas for the transform. In fact, one of our inversions is similar to an inversion formula Kaiser and Streater already obtained in [43], but requires weaker conditions.

Theorem 7.0.5. *Suppose $h \neq 0$ on $(-\infty, 0]$. Then $f \in \mathcal{S}(\mathbb{R}^n)$ can be reconstructed from $P_h f$ as follows:*

$$f(x) = 2^n \pi^{\frac{n-1}{2}} \Gamma(n/2) i \left(\int_0^\infty |\hat{h}(-t)|^2 dt \right)^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}} P_h f(x - vt, v) h'(-t) dt dv.$$

Proof. Clearly, P_h is invariant under a shift with respect to the first n variables. Hence taking the Fourier transform with respect to u looks reasonable. Doing this, we get

$$\widehat{P_h f}(\xi, v) = \hat{f}(\xi) \int_{\mathbb{R}} h(t) e^{itv \cdot \xi} dt = \hat{f}(\xi) \hat{h}(-\xi \cdot v), \quad (7.1)$$

where $\widehat{P_h f}$ is the n -dimensional Fourier transform of $P_h f$ with respect to first n dimensional variable u .

Multiplying (7.1) by $|v|^{-n} \hat{h}(-\xi \cdot v) \xi \cdot v$ and integrating with respect to v yield

$$\begin{aligned} \int_{\mathbb{R}^n} \widehat{P_h f}(\xi, v) \hat{h}(-\xi \cdot v) |v|^{-n} \xi \cdot v dv &= \hat{f}(\xi) \int_0^\infty \int_{S^{n-1}} |\hat{h}(-r\xi \cdot \theta)|^2 \xi \cdot \theta d\theta dr \\ &= |S^{n-2}| \hat{f}(\xi) \int_0^\infty |\xi| \int_{-1}^1 |\hat{h}(-r|\xi|t)|^2 (1-t^2)^{(n-3)/2} t dt dr \\ &= |S^{n-2}| \hat{f}(\xi) \int_0^\infty |\hat{h}(-r)|^2 dr \int_{-1}^1 (1-t^2)^{(n-3)/2} dt \\ &= 2\pi^{(n+1)/2} \Gamma(n/2)^{-1} \hat{f}(\xi) \int_0^\infty |\hat{h}(-t)|^2 dt. \end{aligned} \quad (7.2)$$

where in the first line, we switched from the Cartesian coordinate $v \in \mathbb{R}^n$ to the polar coordinates $(r, \theta) \in [0, \infty) \times S^{n-1}$ and in the third line, we used the Fubini-Tonelli theorem and changed the variable r to $r/|\xi|t$. We also use the known relation

$$\int_{S^{n-1}} f(\omega \cdot \theta) d\theta = |S^{n-2}| \int_{-1}^1 f(t) (1-t^2)^{(n-3)/2} dt, \quad (7.3)$$

for any integrable function f on \mathbb{R} and $\omega \in S^{n-1}$ [54].

In the right hand side of (7.2), $\int_0^\infty |\hat{h}(-t)|^2 dt$ is always positive. Thus we have

$$\begin{aligned}\hat{f}(\xi) &= 2^{-1}\pi^{-(n+1)/2}\Gamma(n/2) \left(\int_0^\infty |\hat{h}(-t)|^2 dt \right)^{-1} \int_{\mathbb{R}^n} \widehat{P_h f}(\xi, v) \hat{h}(-\xi \cdot v) |v|^{-n} \xi \cdot v dv \\ &= 2^{-1}i\pi^{-(n+1)/2}\Gamma(n/2) \left(\int_0^\infty |\hat{h}(-t)|^2 dt \right)^{-1} \int_{\mathbb{R}^n} \widehat{P_h f}(\xi, v) \hat{h}'(-\xi \cdot v) |v|^{-n} dv.\end{aligned}$$

□

Remark 7.0.3. Suppose $h \neq 0$ on $[0, \infty)$. Then similarly to (7.2), we have

$$\int_{\mathbb{R}^n} \widehat{P_h f}(\xi, -v) \hat{h}(\xi \cdot v) |v|^{-n} \xi \cdot v dv = 2\pi^{(n+1)/2}\Gamma(n/2)^{-1} \hat{f}(\xi) \int_0^\infty |\hat{h}(t)|^2 dt. \quad (7.4)$$

Using (7.4) instead of (7.2), we get a similar inversion.

Remark 7.0.4. This inversion is similar to that of [43]. We, however, multiply (7.1) by $|v|^{-n} \hat{h}(-\xi \cdot v)(\xi \cdot v)$, unlike the factor $|v|^{-n} \hat{h}(-\xi \cdot v)$ in [43]. This makes it unnecessary to require that h is admissible ($\hat{h}(0) = 0$).

Now we present another inversion formula.

Theorem 7.0.6. If $h \neq 0$ on $(-\infty, 0]$, then we have for $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}f(x) &= C_n \int_{S^{n-1}} \int_0^\infty \int_{\theta^\perp} \int_{\theta \cdot x}^\infty P_h f(u + \tau\theta, r\theta) h^{(n)} \left(\frac{\theta \cdot x - \tau}{r} \right) r^{-n-1} d\tau du dr d\theta, \\ C_n &= (2\pi)^{-n+1} (-i)^n \left(\int_0^\infty |\hat{h}(-t)|^2 dt \right)^{-1},\end{aligned}$$

where $h^{(n)}$ is the n -th derivative of h .

Proof. Let us consider $P_h f(u + \tau v, v)$ for $u \cdot v = 0$ and $\tau \in \mathbb{R}$. Then we have

$$P_h f(u + \tau v, v) = \int_{\mathbb{R}} f(u + \tau v + tv) h(t) dt = \int_{\mathbb{R}} f(u + tv) h(t - \tau) dt.$$

Switching from the Cartesian coordinate $v \in \mathbb{R}^n$ to the polar coordinates $(r, \theta) \in [0, \infty) \times S^{n-1}$, we get

$$P_h f(u + \tau r \theta, r \theta) = \int_{\mathbb{R}} f(u + tr \theta) h(t - \tau) dt.$$

Then P_h is invariant under a shift with respect to τ . Taking the Fourier transform with respect to τ looks reasonable. To get $\hat{f}(\sigma \theta)$, we take the Fourier transform with respect to τ and integrate with respect to $u \in \theta^\perp$ so that

$$\begin{aligned} \widehat{P_h f}(\sigma/r\theta, r\theta) &= r \int_{\theta^\perp} \int_{\mathbb{R}} P_h f(u + \tau r \theta, r \theta) e^{-i\sigma \tau} d\tau du \\ &= r \int_{\theta^\perp} \int_{\mathbb{R}} f(u + tr \theta) e^{-i\sigma t} dt du \hat{h}(-\sigma) \\ &= \int_{\theta^\perp} \hat{f}(u + \sigma/r\theta) du \hat{h}(-\sigma) = \hat{f}(\sigma/r\theta) \hat{h}(-\sigma), \end{aligned}$$

or

$$\widehat{P_h f}(-\sigma \theta, r \theta) = \hat{f}(\sigma \theta) \hat{h}(-r \sigma). \quad (7.5)$$

Multiplying by $\hat{h}(-r \sigma)$ and integrating (7.5) with respect to r yield

$$\int_0^\infty \widehat{P_h f}(\sigma \theta, r \theta) \hat{h}(-r \sigma) dr = \hat{f}(\sigma \theta) \int_0^\infty |\hat{h}(-r \sigma)|^2 dr = \hat{f}(\sigma \theta) \sigma^{-1} \int_0^\infty |\hat{h}(-r)|^2 dr. \quad (7.6)$$

Similarly to Theorem 7.0.5, we have

$$\hat{f}(\sigma\theta) = \left(\int_0^\infty |\hat{h}(-t)|^2 dt \right)^{-1} \sigma \int_0^\infty \widehat{P_h f}(\sigma\theta, r\theta) \hat{h}(r\sigma) dr,$$

or

$$f(x) = (2\pi)^{-n} \left(\int_0^\infty |\hat{h}(-t)|^2 dt \right)^{-1} \int_{S^{n-1}} \int_0^\infty \left(\int_0^\infty \widehat{P_h f}(\sigma\theta, r\theta) \hat{h}(r\sigma) dr \right) e^{i\sigma\theta \cdot x} \sigma^n d\sigma d\theta. \quad (7.7)$$

Consider the inner integral in (7.7). Then we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \widehat{P_h f}(\sigma\theta, r\theta) \hat{h}(r\sigma) e^{i\sigma\theta \cdot x} \sigma^n dr d\sigma \\ &= \int_0^\infty \int_{\theta^\perp} \int_{\mathbb{R}} P_h f(u + \tau\theta, r\theta) \int_0^\infty \hat{h}(r\sigma) e^{i\sigma(\theta \cdot x - \tau)} \sigma^n d\sigma d\tau dudr \\ &= (-i)^n \int_0^\infty \int_{\theta^\perp} \int_{\mathbb{R}} P_h f(u + \tau\theta, r\theta) r^{-n-1} \int_0^\infty \widehat{h^{(n)}}(\sigma) e^{i\sigma(\theta \cdot x - \tau)/r} d\sigma d\tau dudr, \end{aligned}$$

where in the last line, we changed variables $\sigma \rightarrow \sigma/r$. □

Remark 7.0.5. *Similarly to (7.6), we have*

$$\int_0^\infty \widehat{P_h f}(-\sigma\theta, r\theta) \hat{h}(r\sigma) dr = \hat{f}(-\sigma\theta) |\sigma|^{-1} \int_0^\infty |\hat{h}(t)|^2 dt. \quad (7.8)$$

If $h \neq 0$ on $[0, \infty)$, we have a similar inversion as in remark 7.0.3.

Theorem 7.0.7. *Let h be non-vanishing at a point $a \in \mathbb{R}$. For $f \in \mathcal{S}(\mathbb{R}^n)$, we have for $u = (u_1, u') \in \mathbb{R} \times \mathbb{R}^{n-1}$*

$$\sigma \widehat{P_h f}(\sigma, u', a\sigma, v') = 2\pi \hat{f}(\sigma, av' + u') h(a).$$

Here \hat{f} is the 1-dimensional Fourier transform of f with respect to the first variable x_1 and $\widehat{P_h f}$ is the 2-dimensional Fourier transform of $P_h f$ with respect to the variables (u_1, v_1) .

Proof. Taking the Fourier transform of $P_h f(u, v)$ with respect to u_1 yields

$$\int_{\mathbb{R}} P_h f(u, v) e^{-i\sigma u_1} du_1 = \int_{\mathbb{R}} \hat{f}(\sigma, u' + tv') e^{itv_1\sigma} h(t) dt.$$

To get \hat{f} , multiplying $e^{-iav_1\sigma}$ and integrating with respect to v_1 give

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} P_h f(u, v) e^{-i(av_1 + u_1)\sigma} du_1 dv_1 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\sigma, tv' + u') e^{itv_1\sigma} h(t) e^{-iav_1\sigma} dt dv_1 \\ &= \int_{\mathbb{R}} \hat{f}(\sigma, tv' + u') h(t) \int_{\mathbb{R}} e^{i(t-a)v_1\sigma} dv_1 dt \\ &= 2\pi \int_{\mathbb{R}} \hat{f}(\sigma, tv' + u') h(t) \frac{\delta(t-a)}{\sigma} dt \\ &= 2\pi \hat{f}(\sigma, av' + u') h(a) \sigma^{-1}. \end{aligned}$$

□

Remark 7.0.6. Theorem 7.0.7 leads naturally to a Fourier type inversion formula, supplementing the inverse Fourier transform.

Remark 7.0.7. Even if the domain of u is restricted to a line, say x_1 -axis, then we get the analogue of Theorem 7.0.7, i.e., for $a \in \mathbb{R}$ with $h(a) \neq 0$,

$$\sigma \widehat{P_h f}(\sigma, a\sigma, v') = 2\pi \hat{f}(\sigma, av') h(a),$$

so we can still reconstruct f from $P_h f$.

When $n = 2$, we can get a series formula for an inversion of the windowed ray transform, by using circular harmonics. Consider $P_h f(u, u^\perp)$ where $u^\perp = (-u_2, u_1)$.

Let $g(\rho, \theta)$ be the function $P_h f(u, u^\perp)$ where $\rho = |u|$ and $\theta = u/|u|$, and let $f(r, \phi)$ be the image function in polar coordinates. Then the Fourier series of f and g with respect to their angular variables can be written as

$$f(r, \phi) = \sum_{l=-\infty}^{\infty} f_l(r) e^{il\phi}, \quad g(\rho, \theta) = \sum_{l=-\infty}^{\infty} g_l(\rho) e^{il\theta},$$

where the Fourier coefficients are given by

$$f_l(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \phi) e^{-il\phi} d\phi, \quad g_l(\rho) = \frac{1}{2\pi} \int_0^{2\pi} g(\rho, \theta) e^{-il\theta} d\theta.$$

Theorem 7.0.8. *Let $f \in C_c^\infty(\mathbb{R}^2)$. If $h \in C_c^\infty(\mathbb{R})$ is not odd, then we have*

$$\mathcal{M}g_l(s) = \mathcal{M}f_l(s+1)\mathcal{M}H(s), \quad (7.9)$$

where

$$H(r) = \begin{cases} \left[h\left(\frac{1}{r}\sqrt{1-\frac{1}{r^2}}\right) + h\left(-\frac{1}{r}\sqrt{1-\frac{1}{r^2}}\right) \right] \frac{e^{il \arccos r}}{\sqrt{1-r^2}} & \text{if } r < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We can write $P_h f(u, u^\perp)$ as

$$\int_{\mathbb{R}} f(u + tu^\perp) h(t) dt = \int_{\mathbb{R}^2} f(x) h\left(\frac{x \cdot u^\perp}{|u|^2}\right) \delta\left(|u| - x \cdot \frac{u}{|u|}\right) dx.$$

Then we have

$$\begin{aligned}
g_l(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} g(\rho, \theta) e^{-il\theta} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} f(x) h\left(\frac{x \cdot (-\sin \theta, \cos \theta)}{\rho}\right) \delta(\rho - x \cdot (\cos \theta, \sin \theta)) dx e^{-il\theta} d\theta.
\end{aligned}$$

Changing variables $x \rightarrow (r, \phi) \in [0, \infty) \times [0, 2\pi)$ gives that $g_l(\rho)$ is equal to

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} f(r(\cos \phi, \sin \phi)) h\left(\frac{r \sin(\phi - \theta)}{\rho}\right) \delta(\rho - r \cos(\phi - \theta)) r e^{-il\theta} dr d\phi d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} f(r(\cos \phi, \sin \phi)) h\left(\frac{r \sin(\phi - \theta)}{\rho}\right) \delta(\rho - r \cos(\phi - \theta)) r e^{-il\theta} dr d\phi d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} f(r(\cos \phi, \sin \phi)) h\left(-\frac{r \sin \theta}{\rho}\right) \delta(\rho - r \cos \theta) r e^{-il\theta - i\phi} dr d\phi d\theta,
\end{aligned}$$

where in the last line, we changed variables $\theta \rightarrow \theta + \phi$. Continuing to compute g_l , we get

$$\begin{aligned}
g_l(\rho) &= \int_0^\infty \int_0^{2\pi} f_l(r) h\left(\frac{r \sin \theta}{\rho}\right) \delta(\rho - r \cos \theta) r e^{il\theta} dr d\theta \\
&= \int_\rho^\infty f_l(r) \left[h\left(\frac{r}{\rho} \sqrt{1 - \left(\frac{r}{\rho}\right)^2}\right) + h\left(-\frac{r}{\rho} \sqrt{1 - \left(\frac{r}{\rho}\right)^2}\right) \right] \frac{r e^{il \arccos \frac{\rho}{r}}}{\sqrt{r^2 - \rho^2}} dr \\
&= (r f_l(r)) \times H(\rho),
\end{aligned}$$

where

$$f \times H(r) = \int_0^\infty f(s) H\left(\frac{r}{s}\right) \frac{ds}{s}.$$

Taking the Mellin transform \mathcal{M} of g_l and using property (1.3) complete the proof. □

Corollary 7.0.3. *Let $f_l(r)$ be the l -th Fourier coefficient of the twice differentiable function f with compact support. Then for any $t > 1$ we have*

$$f_l(r) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{t-Ti}^{t+Ti} r^{-s} \frac{\mathcal{M}g_l(s-1)}{\mathcal{M}H(s-1)} ds. \quad (7.10)$$

Proof. For $a > 1$ and $b \in \mathbb{R}$, $|\mathcal{M}f_l(a+bi)|$ is finite because

$$\int_0^\infty r^{a+bi-1} |f_l(r)| dr \leq C \int_0^R r^{a-1} |e^{ib \ln r}| dr,$$

where C is the upper bound of $|f_l|$ and R is radius of a ball containing $\text{supp } f$. Thus, $\mathcal{M}f_l(s)$ is analytic on $\{z \in \mathbb{C} : \Re z > 1\}$. Integrating by parts twice, we get

$$\mathcal{M}f_l(s) = \int_0^\infty f_l''(r) \frac{r^{s+1}}{s(s+1)} d\rho,$$

which implies $\mathcal{M}f_l(s) = O(s^2)$. Hence $\mathcal{M}f_l(t+si)$ is integrable and we can apply the inverse Mellin transform [25, 70] which gives formula (7.10). \square

8. CONCLUSION

In this work, we studied a variety of integral-geometric transforms arising in various types of medical, industrial, and homeland security imaging: radio tomography, single scatter optical tomography, thermo-/photo-acoustic tomography, and Compton camera imaging.

New inversion formulas, stability estimates, and range conditions are established.

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APPENDIX A

PUBLICATIONS

Some parts of the dissertation are presented in

1. Sunghwan Moon. On the determination of a function from an elliptical Radon transform, ArXiv e-prints:1302.4396
2. Gaik Ambartsoumian and Sunghwan Moon. A series formula for inversion of the V-line Radon transform in a disc, to appear in Computers and Mathematics with Applications.

It is planned to prepare the remaining parts for submission.

The following was not included into the dissertation.

- Jin Hong and Sunghwan Moon, A comparison of cryptanalytic tradeoff algorithm, Journal of Cryptology, 26(4):559-637, 2013.

APPENDIX B

CONFERENCE AND SCHOOL

Here is the list of conferences and schools which I attended.

1. Introduction to the Mathematics of Seismic Imaging, MSRI, Jul. 29-Aug. 9, 2013.
2. CBMS Conference ‘Mathematical Methods of Computed Tomography’, UT Arlington, May 28-Jun. 2, 2012.
3. Geometric Analysis on Euclidean and Homogeneous Spaces, Tufts University, Jan. 7-9, 2012.
4. Joint Mathematics Meetings, Boston, MA, Jan. 4-7, 2012.
5. Inverse Problems Conference, TAMU, Oct. 1, 2011.
6. Applied Inverse Problems Conference, TAMU, May 23-27, 2011.
7. Group Actions on Measure Spaces, TAMU, Mar. 24-27, 2011.
8. Waves and Spectra, TAMU, Jan. 11-14, 2011.
9. Texas Geometry and Topology Conference, TAMU, Nov. 12-14, 2010.